Spring 2002

EEE598D: Analog Filter & Signal Processing Circuits

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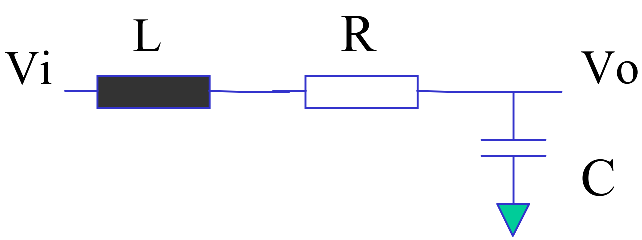
Today: Continuous-Time Filter Fundamental

- Fourier Transformation
- Laplace Transformation
- Transfer Function (TF)

RLC Network and Response

- The traditional question:

  For a given input $v_i(t)$ signal and a RLC network, what is the output signal $v_o(t)$?



$$\begin{cases} LC\dfrac{d^2v_o(t)}{dt^2} + RC\dfrac{dv_o(t)}{dt} + v_o(t) = v_i(t) \\ v_o(t)\big|_{t=0} = v_o(0) \\ \dfrac{dv_o(t)}{dt}\Big|_{t=0} = \dfrac{dv_o(0)}{dt} \end{cases} \qquad (1.1)$$

- The solution:

  It is a differential equation.

  - Solvable but not trivial!!

Steady State Response

- Sometimes we only interest in the steady state responses of the form:

$$\begin{cases} v_i(t) = V_i e^{j\omega t} \, and \\ v_o(t) = V_o e^{j\omega t} \end{cases} \qquad (1.2)$$

- Equation (1.1) can be simplified as:

$$(1 - LC\omega^2 + j\omega RC)V_o = V_i \qquad or$$

$$H(j\omega) = \frac{V_o}{V_i} = \frac{1}{1 - LC\omega^2 + j\omega RC} \qquad (1.3)$$

Frequency Response

- The magnitude (gain) response of the network is

$$\left|H(j\omega)\right| = \frac{1}{\sqrt{(1-LC\omega^2)^2+(RC\omega)^2}} \qquad (1.5)$$

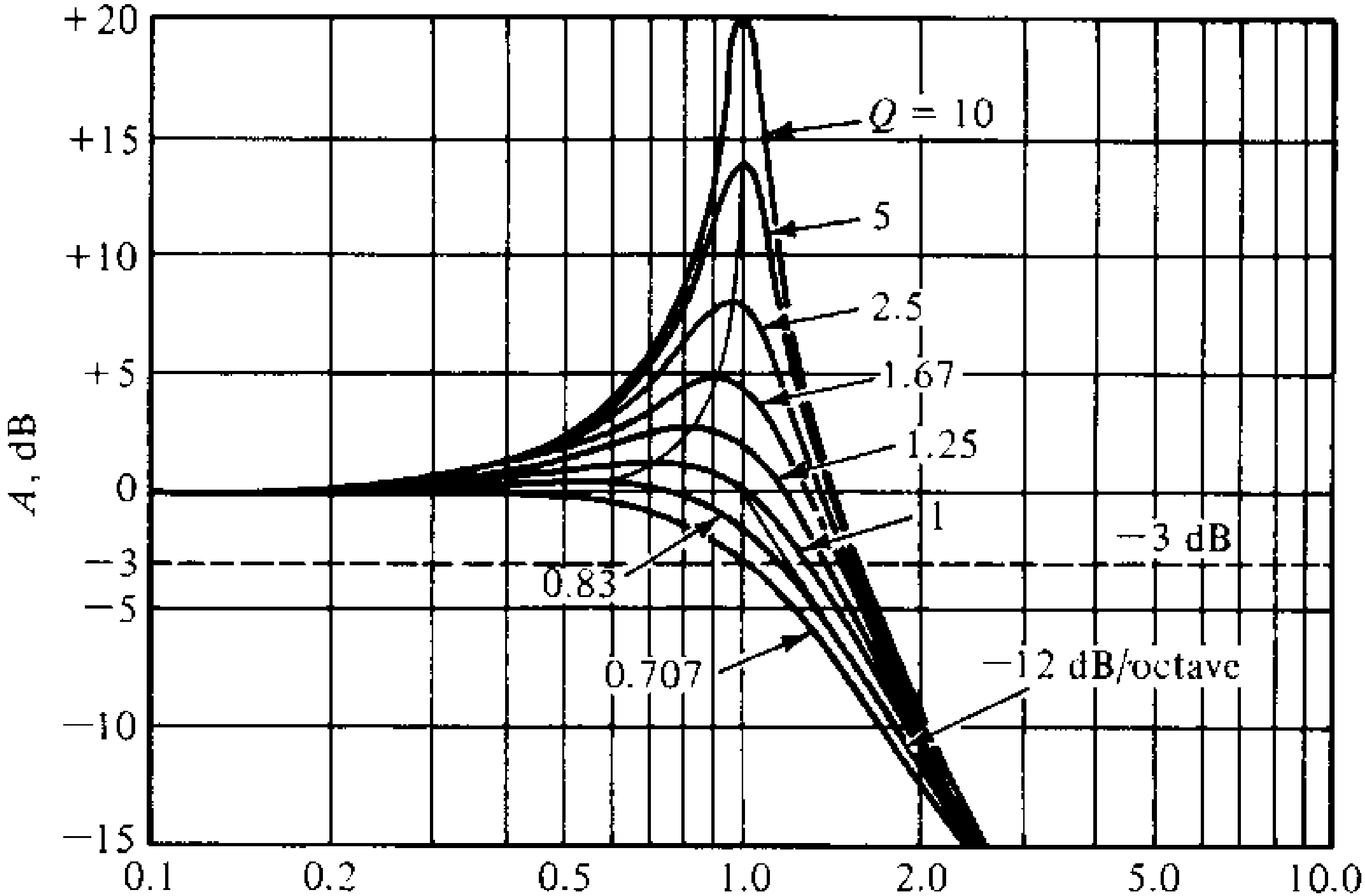
- We have:

$$\left|H(j\omega)\right|_{\omega\to 0} = 1$$

$$\left|H(j\omega)\right|_{\omega\to\infty} = 1/\omega^2\,\Big|_{\omega\to\infty} = 0$$

$$\left|H(j\omega)\right|_{\omega\to 1/\sqrt{LC}} = \frac{1}{R}\sqrt{\frac{L}{C}} \equiv Q$$

Gain Response



$$\omega\sqrt{LC}$$

$$Q \equiv \frac{1}{R}\sqrt{\frac{L}{C}}$$

Frequency Response

- The phase response of the network is

$$\theta(j\omega) = \tan^{-1}\left(\frac{RC\omega}{(1-LC\omega^2)}\right) \qquad (1.6)$$

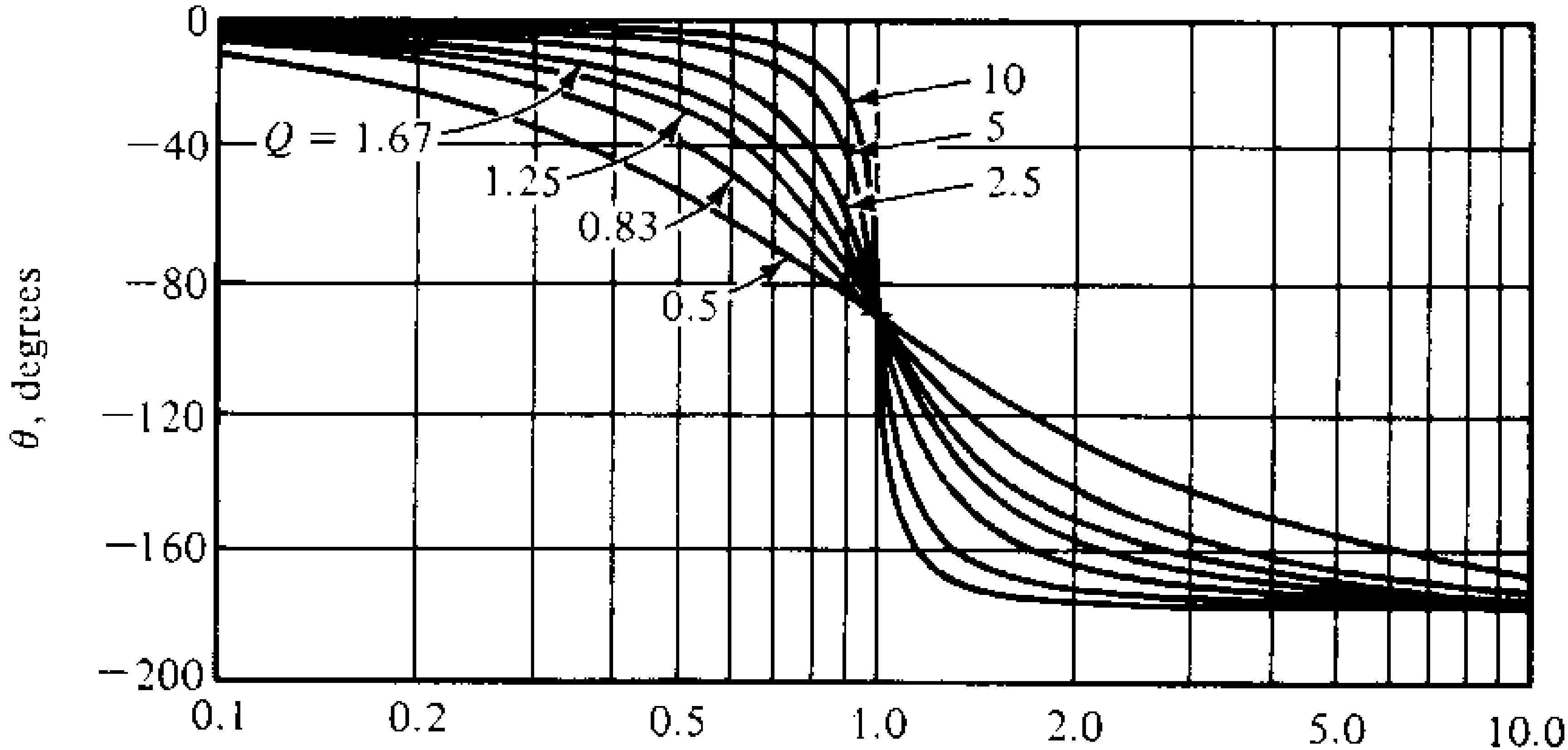
- We have:

$$\theta(j\omega)|_{\omega\to 0} = 0$$

$$\theta(j\omega)|_{\omega\to 1/\sqrt{LC}} = -\pi/2$$

$$\theta(j\omega)|_{\omega\to\infty} = -\pi$$

Phase Response



$\omega\sqrt{LC}$

General Network Responses

- Question 1: How is the network response to a general input (non-sinusoidal) signal?

- Question 2: What is the network’s transient response (non-steady-state response)?

Answer to Q1: Fourier Transformation

- A generic input signal can be expressed by combination of sinusoidal signal using Fourier transformation

The Fourier Transformation

- The Fourier Transform pair is defined as

$$\left\{ \begin{aligned} F(j\omega) &= \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \\ f(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(j\omega) e^{j\omega t} dt \end{aligned} \right. \tag{1.7}$$

Spectrum of f(t)

Useful *Fourier* Transform Pairs

$$
\begin{array}{lcl}
f(t) & & F(j\omega) \\
df(t)/dt & & j\omega F(j\omega) \\
\int_{-\infty}^{t} f(\tau)\, d\tau & & \frac{1}{j\omega} F(j\omega) + \pi F(0)\delta(\omega) \\
f_1(t) * f_2(t) & \Leftrightarrow & F_1(j\omega) F_2(j\omega) \\
\int_{-\infty}^{\infty} |f(t)|^2\, dt & & \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(j\omega)|^2\, d\omega \\
f(t - t_0) & & F(j\omega) e^{-j\omega t_0} \\
f(t) e^{j\omega_0 t} & & F(j\omega - j\omega_0) \\
k_1 f_1(t) + k_2 f_2(t) & & k_1 F_1(j\omega) + k_2 F_2(j\omega)
\end{array}
\quad (1.8)
$$

Useful *Fourier* Transform Pairs

$\delta(t)$		1
$u(t)$		$\pi\delta(\omega) + \dfrac{1}{j\omega}$
$e^{a\|t\|}$		$\dfrac{2a}{\omega^2 + a^2}$
$\begin{rcases} 1, & \|t\| < t_0 \\ 0, & \|t\| > t_0 \end{rcases}$	$\Leftrightarrow$	$(2/\omega)\sin \omega t_0$
$\dfrac{\sin \omega_0 t}{\pi t}$		$\begin{cases} 1, & \|\omega\| < \omega_0 \\ 0, & \|\omega\| > \omega_0 \end{cases}$
$e^{j\omega_0 t}$		$2\pi\delta(\omega - \omega_0)$
1		$2\pi\delta(\omega)$
$\cos \omega_0 t$		$\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$
$\sin \omega_0 t$		$-j\pi\delta(\omega - \omega_0) + j\pi\delta(\omega + \omega_0)$

(1.9)

Response by Fourier Transformation

- Apply Fourier Transform to (1.1), we have

$$-LC\omega^2 V_o(j\omega) + j\omega RC V_o(j\omega) + V_o(j\omega) = V_i(j\omega)$$

$$\Rightarrow H(j\omega) \equiv \frac{V_o(j\omega)}{V_i(j\omega)} = \frac{1}{1 - LC\omega^2 + j\omega RC}$$

Answer to Q2: Laplace Transformation

- The transient response of the network can be calculated by using the Lapalce Transformation

Laplace Transformation

- *The (one side) Laplace* Transform is defined as:

$$\begin{cases} F(s) \equiv L(f(t)) \equiv \int\limits_{0}^{\infty} f(t)e^{-st}dt \\ \\ f(t) = L^{-1}(F(s)) = \dfrac{1}{2\pi j} \int\limits_{\sigma - j\infty}^{\sigma + j\infty} F(s)e^{-st}dt \end{cases} \qquad (1.10)$$

Useful *Laplace* Transform Pairs

$$
\begin{aligned}
&f(t) && F(s) \\
&df(t)/dt && sF(s) - f(0) \\
&\int_0^t f(\tau)\, d\tau && \frac{1}{s} F(s) \\
&f_1(t) * f_2(t) \quad \Leftrightarrow && F_1(s) F_2(s) \qquad (1.11) \\
&f(t-a)u(t-a),\ a \geq 0 && e^{-as} F(s) \\
&tf(t) && -\frac{dF(s)}{ds} \\
&e^{at} f(t) && F(s-a) \\
&k_1 f_1(t) + k_2 f_2(t) && k_1 F_1(s) + k_2 F_2(s)
\end{aligned}
$$

Useful *Laplace* Transform Pairs

$$
\begin{aligned}
\delta(t) &\Leftrightarrow 1 \\
K \text{ or } Ku(t) &\Leftrightarrow \frac{K}{s} \\
t^n &\Leftrightarrow \frac{n!}{s^{n+1}} \\
e^{at} &\Leftrightarrow \frac{1}{s-a} \\
e^{at}\cos bt &\Leftrightarrow \frac{s-a}{(s-a)^2+b^2} \\
e^{at}\sin bt &\Leftrightarrow \frac{b}{(s-a)^2+b^2}
\end{aligned}
\qquad (1.12)
$$

General Continuous-Time Linear Filter

- A general continuous-time (CT) linear filter can be expressed as a differential equation:

$$
\begin{cases}
\displaystyle\sum_{k=0}^{n} b_k \frac{d^k v_o(t)}{dt^k} = \sum_{k=0}^{m} a_k \frac{d^k v_i(t)}{dt^k} \\[2ex]
\displaystyle\frac{d^k v_o(t)}{dt^k}\Big|_{t=0} = \frac{d^k v_o(0)}{dt^k} \qquad k = 0,1,\ldots n-1
\end{cases}
\tag{1.13}
$$

Filter Response using *Laplace* Transform

- The response of the above filter can be solved using Laplace Transform:

$$L\left(\sum_{k=0}^{n} b_k \frac{d^k v_o(t)}{dt^k}\right) = L\left(\sum_{k=0}^{m} a_k \frac{d^k v_i(t)}{dt^k}\right)$$

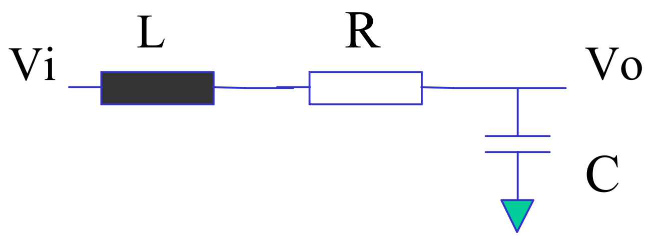
$$V_o(s) = \frac{\sum_{k=0}^{m} a_k s^k}{\sum_{k=0}^{n} b_k s^k} V_i(s) + T(s) \qquad (1.14)$$

Initial condition

$$v_o(t) = L^{-1}\left[\left(\frac{\sum_{k=0}^{m} a_k s^k}{\sum_{k=0}^{n} b_k s^k}\right) V_i(s)\right] + L^{-1}[T(s)] \qquad (1.15)$$

Example of *Laplace* Transform

- Look at filter response of the above filter for given initial condition



$$\begin{cases} LC\dfrac{d^2 v_o(t)}{dt^2} + RC\dfrac{dv_o(t)}{dt} + v_o(t) = Ae^{-j\omega t} \\ v_o(t)\,|_{t=0} = 0 \\ \dfrac{dv_o(t)}{dt}\,|_{t=0} = 0 \end{cases} \qquad (1.16)$$

Example of *Laplace* Transform

- Using (1.5), we have

$$V_o(s) = \frac{1}{(LCs^2 + RCs + 1)} \frac{V_m}{(s + j\omega)} \qquad (1.17)$$

- Let

$$\begin{cases} \alpha \equiv \dfrac{R}{2L} \\ \beta \equiv \sqrt{\dfrac{1}{LC} - (\dfrac{R}{2L})^2} \end{cases} \qquad (1.18)$$

- We have

$$V_o(s) = \frac{1}{(s + \alpha + j\beta)(s + \alpha - j\beta)} \frac{V_m}{(s + j\omega)} \qquad (1.19)$$

Example of *Laplace* Transform

- With partial-fraction expansion, we have

$$V_o(s) = \frac{A}{(s + j\omega)} + \frac{B}{(s + \alpha + j\beta)} + \frac{C}{(s + \alpha - j\beta)} \quad (1.20)$$

- where

$$\left\{ \begin{aligned} A &= \frac{V_m}{s^2 LC + sRC + 1} \Big|_{s = j\omega} \\ B &= \frac{V_m}{(s + \alpha - j\beta)(s + j\omega)} \Big|_{s = \alpha + j\beta} \\ C &= \frac{V_m}{(s + \alpha + j\beta)(s + j\omega)} \Big|_{s = \alpha - j\beta} \end{aligned} \right. \quad (1.21)$$

Example of *Laplace* Transform

- Using (1.3) the response can be solved as:

$$v_i(t) = Ae^{-j\omega t} + \underbrace{Be^{-\alpha t - j\beta t} + Ce^{-\alpha t + j\beta t}}_{\text{transient response}} \quad (1.22)$$

transient response

$(\to 0 \, as \quad t \to \infty)$

$$A = \frac{V_o(s)}{V_i(s)}\Big|_{s=j\omega}$$ steady-state response

Transfer Function

- For steady-state response, the s-domain ratio of the output signal to input signal is known as the transfer function (TF) of the filter:

$$H(s) = \frac{V_o(s)}{V_i(s)} \qquad (1.23)$$

- For the given example above, TF is given as:

$$H(s) = \frac{V_o(s)}{V_i(s)} = \frac{1}{s^2LC + sRC + 1} \qquad (1.24)$$

S-Domain Representation of Operators

- By using the Laplace Transform, we have:

Integrator:

$$y(t) = \int_{t_o}^{t} x(\tau)d\tau \Rightarrow Y(s) = \frac{1}{s}X(s) \Rightarrow \boxed{\int dt \quad \Leftrightarrow \quad \frac{1}{s}} \quad (1.25)$$

Differentiator:

$$y(t) = \frac{dx(t)}{dt} \Rightarrow Y(s) = sX(s) \Rightarrow \boxed{\frac{d}{dt} \quad \Leftrightarrow \quad s} \quad (1.26)$$

Transfer Function

- Generally, for a filter described the following linear differential equation:

$$\sum_{k=0}^{n} b_k \frac{d^k v_o(t)}{dt^k} = \sum_{k=0}^{m} a_k \frac{d^k v_i(t)}{dt^k} \qquad (1.27)$$

- The transfer function can be calculated using *Laplace* Transform as:

$$H(s) = \frac{\sum_{k=0}^{m} a_k s^k}{\sum_{k=0}^{n} b_k s^k} = \frac{a_m s^m + a_{m-1} s^{m-1} + \ldots + a_1 s + a_0}{b_n s^n + b_{n-1} s^{n-1} + \ldots + b_1 s + b_0} \qquad (1.28)$$

S-Domain Impedance of Components

- By Laplace Transform, we have:

Resistor:

$$v(t) = Ri(t) \Rightarrow V(s) = RI(s) \qquad \Rightarrow \qquad Z_R = R$$

Capacitor:

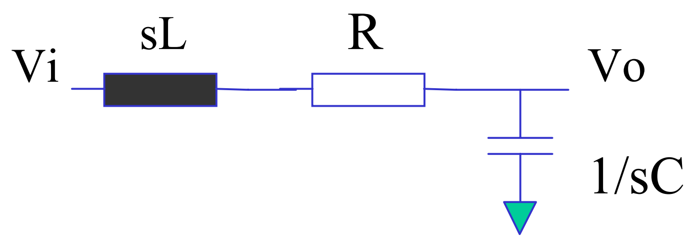
$$\frac{dv(t)}{dt} = \frac{I(t)}{C} \Rightarrow V(s) = \frac{1}{sC} I(s) \qquad \Rightarrow \qquad Z_C = \frac{1}{sC} \qquad (1.29)$$

Inductor:

$$v(t) = L\frac{di(t)}{dt} \Rightarrow V(s) = sLI(s) \qquad \Rightarrow \qquad Z_L = sL$$

Calculation of Transfer Function

- The transfer function of a linear circuit can be calculated by using s-domain impedance through *Kirchhoff's* law:



$$\frac{V_o(s)}{V_i(s)} = \frac{Z_C}{Z_L + Z_R + Z_C} = \frac{1/sC}{sL + R + 1/sC}$$
$$= \frac{1}{s^2LC + sRC + 1} \qquad (1.30)$$

System Model of CT Filters

- Linear Systems

constant

constant

$$K(a_1x_1(t)+a_2x(t)) = a_1K(x_1(t))+a_2K(x_2(t))$$

- Time- (shift-) Invariant Systems

$$\textit{if} \quad y(t)=K(x(t))$$

$$\textit{Then} \quad y(t-t_o)=K(x(t-t_o))$$

S-Domain Representation of Linear Time-Invariant (LTI) Systems

$$L\{y(t)\} = L\{K(x(t))\} = L\{K(\int_{-\infty}^{\infty} x(\zeta)\delta(t-\zeta)d\zeta)\}$$

$$= L\{\int_{-\infty}^{\infty} x(\zeta)K(\delta(t-\zeta))d\zeta\} = L\{x(t)\}\cdot L\{K(\delta(t))\}$$

Let:

$$Y(s) \equiv L\{y(t)\}$$
$$X(s) \equiv L\{x(t)\}$$
$$h(t) \equiv K(\delta(t))$$
$$H(s) \equiv L\{h(t)\}$$

Then:

$$\boxed{\frac{Y(s)}{X(s)} = H(s)}$$

Transfer Function (TF) of system

Impulse response of system

Remark: Continuous-time LTI System can be completely determined by its impulse response.

Steady State Response of Continuous-Time LTI Systems

$$x(t) = Au(t)e^{j\omega t} \quad \Rightarrow X(s) = A\int_0^{\infty} e^{-(s-j\omega)t}dt = \frac{A}{s-j\omega}$$

$$Y(s) = H(s)\cdot X(s) = \frac{H(s)}{s-j\omega} = \frac{H(j\omega)}{s-j\omega} + Other \quad Terms$$

$$y(t)\Big|_{t\to\infty} = Au(t)H(j\omega)e^{j\omega t} + y_h(t) = H(j\omega)x(t)$$

Approach zero for stable system

Steady State Response of the system