Subject: EEE598D Homework#4
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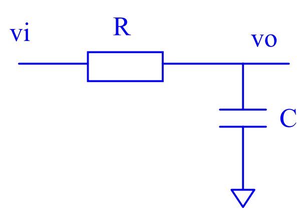
Problem 1 (s-z- transformations)

For the lowpass continuous-time filter shown in Fig. 1, find the z-domain transfer functions H(z) and plot gain and phase responses versus frequency logω (0.0001π/T < ω< π/T) (i.e. BodePlots) under

1) Forward Euler
2) Backward Euler
3) Bilinear , and
4) LDI (or midpoint) transformations



$$H_A(s) \equiv \frac{V_o(s)}{V_i(s)} = \frac{p}{p+s}$$

$$p \equiv \frac{1}{RC}$$

(Assuming 2π/T = 100p).

Solution:
Let

$$\alpha \equiv pT = 2\pi/100 = 0.0628$$

and

$$\Omega \equiv \omega T/\pi$$

in the following analysis

1) For forward Euler transformation: we have

$$\frac{1}{s} = \frac{Tz^{-1}}{1-z^{-1}} \quad \Rightarrow sT = z-1$$

$$H(z) = H_A(s)|_{sT=z-1} = \frac{\alpha}{\alpha + z - 1}$$

Frequency responses are given as

$$H(z)|_{z=e^{j\omega T}} = \frac{\alpha}{\alpha + e^{j\omega T} - 1}$$

at low frequency ($\omega T \ll 1$),

$$e^{j\omega T} \approx 1 + j\omega T$$

the above equation can be simplified as

$$H(z)|_{z=e^{j\omega T}} \approx \frac{\alpha}{\alpha + j\omega T} = \frac{p}{p + j\omega} = H_A(s)|_{s=j\omega}$$

which is the same as the original analog filter response.

In general the filter frequency response can be calculated as

$$H(z)|_{z=e^{j\omega T}} = \frac{\alpha}{\alpha - 1 + \cos(\Omega\pi) + j\sin(\Omega\pi)} = \frac{\alpha}{\sqrt{(\alpha - 1 + \cos(\Omega\pi))^2 + (\sin(\Omega\pi))^2}} e^{-j\tan^{-1}\left(\frac{\sin(\Omega\pi)}{\alpha - 1 + \cos(\Omega\pi)}\right)}$$

2) For backward Euler transformation, we have

$$\frac{1}{s} = \frac{T}{1 - z^{-1}} \qquad \Rightarrow sT = 1 - z^{-1}$$

$$H(z) = H_A(s)|_{sT=1-z^{-1}} = \frac{\alpha}{\alpha + 1 - z^{-1}}$$

Frequency responses are given as

$$H(z)|_{z=e^{j\omega T}} = \frac{\alpha}{\alpha + 1 - e^{-j\omega T}}$$

at low frequency ($\omega T \ll 1$),

$$e^{-j\omega T} \approx 1 - j\omega T$$

the above equation can be simplified as

$$H(z)|_{z=e^{j\omega T}} \approx \frac{\alpha}{\alpha + j\omega T} = \frac{p}{p + j\omega} = H_A(s)|_{s=j\omega}$$

which is the same as the original analog filter response.

In general the filter frequency response can be calculated as

$$H(z)|_{z=e^{j\omega T}} = \frac{\alpha}{\alpha + 1 - \cos(\Omega\pi) + j\sin(\Omega\pi)} = \frac{\alpha}{\sqrt{(\alpha + 1 - \cos(\Omega\pi))^2 + (\sin(\Omega\pi))^2}} e^{-j\tan^{-1}(\frac{\sin(\Omega\pi)}{\alpha + 1 - \cos(\Omega\pi)})}$$

3) For Bilinear transformation, we have

$$\frac{1}{s} = \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}} \qquad \Rightarrow sT = 2\frac{1-z^{-1}}{1+z^{-1}}$$

$$H(z) = H_A(s)|_{sT=1-z^{-1}} = \frac{\alpha}{\alpha + 2\frac{1-z^{-1}}{1+z^{-1}}}$$

Frequency responses are given as

$$H(z)|_{z=e^{j\omega T}} = \frac{\alpha}{\alpha + 2j\tan(\frac{\omega T}{2})}$$

at low frequency ($\omega T \ll 1$),

$$2j\tan(\frac{\omega T}{2}) \approx j\omega T$$

the above equation can be simplified as

$$H(z)|_{z=e^{j\omega T}} \approx \frac{\alpha}{\alpha + j\omega T} = \frac{p}{p + j\omega} = H_A(s)|_{s=j\omega}$$

which is the same as the original analog filter response.

In general the filter frequency response can be calculated as

$$H(z)|_{z=e^{j\omega T}} = \frac{\alpha}{\alpha + 2j\tan(\frac{\Omega\pi}{2})} = \frac{\alpha}{\sqrt{\alpha^2 + \left(2\tan(\frac{\Omega\pi}{2})\right)^2}} e^{-j\tan^{-1}(\frac{2\tan(\frac{\Omega\pi}{2})}{\alpha})}$$

4) For the mid-point transformation: we have

$$\frac{1}{s} = \frac{Tz^{-1/2}}{1 - z^{-1}} \qquad \Rightarrow sT = z^{1/2} - z^{-1/2}$$

$$H(z) = H_A(s)|_{sT=z-1} = \frac{\alpha}{\alpha + z^{1/2} - z^{-1/2}}$$

Frequency responses are given as

$$H(z)|_{z=e^{j\omega T}} = \frac{\alpha}{\alpha + 2j\sin(\frac{\omega T}{2})}$$

at low frequency ($\omega T \ll 1$),

$$2j\sin(\frac{\omega T}{2}) \approx j\omega T$$

the above equation can be simplified as

$$H(z)|_{z=e^{j\omega T}} \approx \frac{\alpha}{\alpha + j\omega T} = \frac{p}{p + j\omega} = H_A(s)|_{s=j\omega}$$

which is the same as the original analog filter response.

In general the filter frequency response can be calculated as

$$H(z)|_{z=e^{j\omega T}} = \frac{\alpha}{\alpha + 2j\sin(\frac{\Omega\pi}{2})} = \frac{\alpha}{\sqrt{\alpha^2 + \left(2\sin(\frac{\Omega\pi}{2})\right)^2}} e^{-j\tan^{-1}\left(\frac{2\sin(\frac{\Omega\pi}{2})}{\alpha}\right)}$$

Bode Plots for all transformations as well as original analog filter: