ASSET PRICES, DEBT CONSTRAINTS AND INEFFICIENCY

GAETANO BLOISE AND PIETRO REICHLIN

Current version: May 3, 2008

Abstract. In this paper, we consider economies with (possibly endogenous) solvency constraints under uncertainty. Long-run inefficiency corresponds to a feasible redistribution yielding a welfare improvement beginning from every contingency reached by the economy. A sort of Cass Criterion (Cass [11]) completely characterizes long-run inefficiency. This criterion involves only observable prices and requires low interest rates in the long-run, exactly as it happens for canonical inefficiency in economies of overlapping generations. In addition, when quantitative limits to liabilities arise from participation constraints, the existence of a feasible welfare improvement, subject to participation, coincides with the introduced notion of long-run inefficiency.

Keywords. Private debt; solvency constraints; default; Cass Criterion; asset prices; incomplete markets; constrained inefficiency.

JEL Classification Numbers. D50, D52, D61, E44, G13.

1. Introduction

Models with debt constraints have been used to explain the time series of output, asset prices and interest rates (Scheinkman and Weiss [26]), to understand and quantify the size of precautionary savings (Aiyagari [2]), to derive the optimal quantity of money (Bewley [7]), or public debt (Woodford [27]), and to prove the existence of asset bubbles (Scheinkman and Weiss [26], Kocherlakota [19], Santos and Woodford [25]). More recently, there has been a great deal of research on the endogenous determination of debt constraints, assuming limited enforcement and incentive constraints (among others, Kehoe and Levine [17], Kocherlakota [20] and Alvarez and Jermann [5]). These studies have reconsidered and quantified the same issues addressed in previous models with exogenous debt limits, as well as other issues, such as debt sustainability under limited contract enforcement (Eaton and Gersovitz [15], Bulow and Rogoff [10], Hellwig and Lorenzoni [16]).

Debt limits prevent the economy from attaining a social optimum because individuals are unable to exploit all trading opportunities and disparities in subjective evaluations of risk might remain at equilibrium. The interesting issue is whether competitive equilibria are constrained optimal, that is, whether there may be benefits from redistributions under the condition that debt constraints cannot be removed. This notion of optimality is particularly relevant when debt limits are endogenous, since, most likely, policy intervention fails in sidestepping the incentive

We are grateful to Paolo Siconolfi, Luca Panaccione and Herakles Polemarchakis for helpful comments and observations. We also gratefully acknowledge financial support from the Italian Ministry of University and Research (PRIN 2006). All remaining misunderstandings, omissions and errors are our own responsibility.

constraints from which debt limits arise. Evidently, the failure of constrained optimality entails some distortions in the intertemporal allocation of resources, which we may define as long-run inefficiency, or lack of transversality. The purpose of this work is to verify whether, at a competitive equilibrium, the mere observation of prices completely reveals the occurrence of long-run inefficiency.

In order to asses welfare implications of debt constraints, we here consider an economy under uncertainty with sequentially complete asset markets and with arbitrarily specified debt constraints, that is, quantitative limits to private liabilities. This specification encompasses not-too-tight debt constraints of Alvarez and Jermann [5] and self-enforcing private debt of Hellwig and Lorenzoni [16], as well as the extreme cases of natural debt limits (Levine and Zame [22], Santos and Woodford [25]) and of no private liabilities or pure borrowing constraints (Bewley [7]). In general, the more severe are debt constraints, the more severe is the lack of complete insurance, or market incompleteness, so amplifying welfare losses. Nevertheless, the understanding of long-run inefficiency admits a unified treatment, independently of the specific nature of debt constraints.

We here define *long-run inefficiency* as the occurrence of a feasible welfare improvement beginning from any contingency, or conditional on any realization of uncertainty along the infinite horizon of the economy. More precisely, given consumption plans of individuals, a feasible reallocation guarantees higher utilities to all individuals conditional on the realization of uncertainty at every period of trade. Thus, long-run inefficiency admits sequential benefits from the redistribution with respect to planned consumptions. Relevantly, this form of inefficiency is independent of the nature of debt constraints.

Some specifications of debt constraints are consistent with well-identified participation constraints. These allows for natural notions of constrained inefficiency, along the lines of Kehoe and Levine [17]. Alvarez and Jermann [5] postulate that debt repudiation induces an exclusion from financial markets, though individuals maintain their labor income; thus, their notion of not-too-tight debt constraints corresponds to participation constraints ensuring that individuals would not benefit, at every contingency, from reverting permanently to autarchy (Kehoe and Levine [17] and Kocherlakota [19]). Hellwig and Lorenzoni [16], instead, assume that unhonored debt deprives from issuing further debt obligations, though individuals might still participate into financial markets for lending (Bulow and Rogoff [10]); their notion of self-enforcing debt coincides with participation constraints guaranteeing that individuals would not profit from repudiating their debt and participating into financial markets subject to pure debt constraints. Furthermore, participation constraints emerge under a variety of assumptions, such as that bankruptcy induces an exclusion from asset markets for a limited number of periods only, or the imposition of collateral requirements (see Phelan [24], Kiyotaki and Moore [18], Lustig [23], Krueger and Uhlig [21] for different formulations of outside options for borrowers). In all these instances, debt constraints are generated by participation constraints, though associated reservation utilities vary according to the hypotheses on the effects of debt repudation. A natural notion of *constrained inefficiency* is given by a feasible welfare improvement subject to participation constraints at reservation utilities sustaining endogenous debt constraints. An hypothetical planner, thus, is restricted by constraints analogous to those inducing market incompleteness. It is worth remarking that, as a matter of mere fact, long-run inefficiency coincides with constrained inefficiency at reservation utilities corresponding to consumption plans of individuals (a sort of *constrained inefficiency at planned consumptions*).

We show that, under acceptably restrictive assumptions, long-run inefficiency of competitive equilibrium is completely identified by a sort of Modified Cass Criterion. Furthermore, constrained inefficiency at given reservation utilities is also fully characterized by the Modified Cass Criterion, provided that debt constraints are consistent with those reservation utilities. Thus, constrained inefficiency is equivalent to long-run inefficiency, that is, to the existence of a feasible redistribution yielding a welfare improvement at every contingency across periods of trade. This is basically the only source of inefficiency conditionally on the fact that market incompleteness cannot be removed.

Long-run inefficiency occurs when *low implied interest rates* prevail at equilibrium. The exact domain of low implied interest rates is captured by the Modified Cass Criterion, which requires the existence of a sequence of bounded positive transfers of commodities, $\{v_t\}$, satisfying, for some ρ in $(0, 1)$,

$$\rho \mathbb{E}_t p_{t+1} v_{t+1} \geq p_t v_t,$$

where $\{p_t\}$ is the sequence of Arrow-Debreu prices, or contingent claims prices. These transfers yield a sequential welfare improvement when distributed, in every period of trade, from unconstrained to constrained individuals. Thus, the Modified Cass Criterion also identifies a class of transfers for policy intervention. The value of the parameter ρ represents an upper bound of the average safe (gross) interest rate prevailing in the long-run. Importantly, inefficiency can be equivalently characterized by means of the positive linear operator defined by

$$T(v)_t = \frac{1}{p_t} \mathbb{E}_t p_{t+1} v_{t+1}.$$

It exactly corresponds to the existence of a real eigenvalue, of such a linear operator, larger than the unity, with associated *bounded* eigenvector. This is reminiscence of the Dominant Root, or Perron-Frobenius, Characterization (Aiyagari and Peled [3]) for recursive equilibria in stochastic overlapping generations economies.

It should be noticed that, according to our findings, *high implied interest rates* (that is, a finitely valued aggregate endowment) are sufficient but not necessary for constrained optimality, in contrast with a claim in Alvarez and Jermann [5]. In fact, under non-stationary endowments, we provide an example of a non-autarchic constrained efficient allocation, according to the notion adopted by these authors, violating high implied interest rates.

An important virtue of any test of long-run inefficiency based on the observation of prices, as the one proposed in this paper, is that, in principle, it is suitable for empirical work. Tests of various versions of the Cass Criterion have been discussed with reference to stochastic overlapping generations economies with production. For example, Abel, Mankiw, Summers and Zeckhauser [1] provide a version of the Cass Criterion called Net Dividend Criterion and they conclude that the US economy is dynamically efficient. However, Barbie, Hagedorn and Kaul [6] point out that a correct implementation of this test requires that net dividends be computed for all potential paths following a given history of states. Consequently, they propose a criterion introduced by Zilcha [28], based on the expected rental rate of capital, and conclude in favor of dynamic efficiency. However, they also stress that, under uncertainty, dynamic efficiency is only a necessary, not a sufficient condition for *interim* Pareto optimality. The absence of capital overaccumulation may still allow for distortions of consumptions across states and time periods.

Difficulties and potentials for testing long-run efficiency in pure exchange overlapping generations economies with uncertainty and sequentially complete (and incomplete) markets are discussed in Chattopadhyay [12] and Bloise and Calciano [9]. The latter contribution considers economies with arbitrarily long, though bounded, horizons for generations and proposes a criterion for inefficiency formally identical to that in this paper. It is suggested that this criterion is more easily applicable in empirical work because of its simpler formulation.

The proposed characterization of constrained inefficiency further clarifies the analogy between economies with debt constraints and economies of overlapping generations. Economies of overlapping generations might exhibit locally indeterminate competitive equilibria and might sustain a positive value of outside money at equilibrium. Similar phenomena emerge in economies with debt constraints. Indeed, debt limits produce a fragmentation of the intertemporal budget constraint, so that impatient individuals do in fact act over a sequence of limited horizons (Bewley [8]), as in overlapping generations economies. The analogy also extends to welfare properties of equilibria. Long-run inefficiency of competitive equilibria can be understood, in both cases, as a failure of the transversality condition and is characterized by lower implied interest rates. Whereas in overlapping generations economies transversality fails because no individual holds a positive fraction of the aggregate endowment, in economies with borrowing constraints transversality is violated because, at equilibrium, no individual is never credit constrained at all date events. We clarify that dynamic efficiency is not restricted to the case of high implied interest rates (*i.e.*, a finitely valued aggregate endowment), but it is also verified when implied interest rates are neither high nor low.

The essay is organized as follows. In section 2, we present the basic assumptions on fundamentals. In sections 3 and 4, we discuss the notion of constrained inefficiency. In section 5, we introduce the notions of equilibrium and of price support. In section 6, we provide the characterization of long-run inefficiency in terms of equilibrium prices. In section 7, we extend the characterization to constrained inefficiency, at reservation utilities consistent with debt constraints, and discuss the relation between our analysis and the analysis of Alvarez and Jermann [5]. Finally, we present an example in appendix A and a complement to Kehoe and Levine's [17] Second Welfare Theorem in appendix B. All proofs are collected in appendix C.

2. Fundamentals

2.1. **Time and uncertainty.** Time and uncertainty are represented by an event-tree $\mathcal{S}$, a countably infinite set, endowed with ordering $\succeq$. For a date-event σ in $\mathcal{S}$, $t(\sigma)$ in $\mathcal{T} = \{0, 1, 2, \ldots, t, \ldots\}$ denotes its date and

$$\sigma_+ = \{\tau \in \mathcal{S}(\sigma) : t(\tau) = t(\sigma) + 1\}$$

is the non-empty finite set of all immediate direct successors, where

$$\mathcal{S}(\sigma) = \{\tau \in \mathcal{S} : \tau \succeq \sigma\}.$$

The initial date-event is ϕ in $\mathcal{S}$, with $t(\phi) = 0$, that is, $\sigma \succeq \phi$ for every σ in $\mathcal{S}$. This construction is canonical (Debreu [14, Chapter 7]).

2.2. **Vector space notation and terminology.** As far as notation and terminology for vector spaces are concerned, we basically follow Aliprantis and Border [4, Chapters 5-8]. Consider the vector space of all real maps on $\mathcal{S}$, $\mathbb{R}^{\mathcal{S}}$, endowed with the canonical (product) ordering. An element v of $\mathbb{R}^{\mathcal{S}}$ is positive (respectively, strictly positive) if $v_\sigma \geq 0$ for every σ in $\mathcal{S}$ (respectively, $v_\sigma > 0$ for every σ in $\mathcal{S}$). For an element v of $\mathbb{R}^{\mathcal{S}}$, v^+ in $\mathbb{R}^{\mathcal{S}}$ and v^- in $\mathbb{R}^{\mathcal{S}}$ are, respectively, its positive part and its negative part, so that $v = v^+ - v^-$ in $\mathbb{R}^{\mathcal{S}}$ and $|v| = v^+ + v^-$ in $\mathbb{R}^{\mathcal{S}}$. Also, for an arbitrary collection $\{v^j\}_{j \in \mathcal{J}}$ of elements of $\mathbb{R}^{\mathcal{S}}$, its supremum and its infimum in $\mathbb{R}^{\mathcal{S}}$, if they exist, are denoted, respectively, by

$$\bigvee_{j \in \mathcal{J}} v^j \text{ and } \bigwedge_{j \in \mathcal{J}} v^j.$$

Finally, the positive cone of any (Riesz) vector subspace F of $\mathbb{R}^{\mathcal{S}}$ is $\{v \in F : v \geq 0\}$.

2.3. **Commodity space.** There exists a single commodity that is traded and consumed at every date-event. The *commodity space* is L, the (Riesz) vector space of all *bounded* real maps on $\mathcal{S}$. The vector space L is endowed with the supremum norm given by

$$\|v\| = \inf \{\lambda > 0 : |v| \leq \lambda u\},$$

where here u denotes the unit of L. A linear functional f on L is strictly positive if, for every non-null positive element v of L, $f \cdot v > 0$. It is order-continuous if, for every element v of L,

$$f \cdot v = \sum_{\sigma \in \mathcal{S}} f \cdot v_\sigma,$$

where we use the decomposition $\mathbb{R}^{\mathcal{S}} = \oplus_{\sigma \in \mathcal{S}} \mathbb{R}_\sigma$.

2.4. **Preferences.** There is a finite set $\mathcal{J}$ of individuals. For every individual i in $\mathcal{J}$, the *consumption space* X^i is the positive cone of L. Though more general preferences can be encompassed in our analysis, it simplifies to assume time additively separable utilities. Preferences of individual i in $\mathcal{J}$ on X^i are represented by

$$U^i\left(x^i\right) = \sum_{\sigma \in \mathcal{S}} \pi^i_\sigma u^i\left(x^i_\sigma\right),$$

where π^i is a strictly positive order-continuous linear functional on L and $u^i : \mathbb{R}_+ \to \mathbb{R}$ is a bounded, smooth, smoothly strictly increasing and smoothly strictly concave per-period utility function. For every date-event σ in $\mathcal{S}$, let

$$U^i_\sigma\left(x^i\right) = \frac{1}{\pi^i_\sigma} \sum_{\tau \in \mathcal{S}(\sigma)} \pi^i_\tau u^i\left(x^i_\tau\right),$$

so that

$$U^i_\sigma\left(x^i\right) = u^i\left(x^i_\sigma\right) + \frac{1}{\pi^i_\sigma} \sum_{\tau \in \sigma_+} \pi^i_\tau U^i_\tau\left(x^i\right).$$

Finally, we assume that there exists a sufficiently small $1 > \eta > 0$ satisfying, for every individual i in $\mathcal{J}$, at every σ in $\mathcal{S}$,

$$\text{(UI)} \qquad \pi^i_\sigma \geq \eta \sum_{\sigma \in \mathcal{S}(\sigma)} \pi^i_\tau.$$

This hypothesis imposes *uniform impatience* across individuals at interior consumption plans.

2.5. **Allocation.** An *allocation* x is an element of $X = \prod_{i \in \mathcal{J}} X^i$. An allocation x in X is interior if there exists $\lambda > 0$ such that

$$\bigwedge_{i \in \mathcal{J}} x^i \geq \lambda u,$$

where here u denotes the unit of L. The hypothesis of interiority is stronger than necessary and is maintained only to simplify presentation.

2.6. **Subjective prices.** For an individual i in $\mathcal{J}$, at an interior consumption plan x^i in X^i, the *subjective price* is p^i is an element of P^i, the set of all strictly positive order-continuous linear functional on L, satisfying, at every consumption plan z^i in X^i,

$$U^i\left(z^i\right) - U^i\left(x^i\right) \leq \sum_{\sigma \in \mathcal{S}} p^i_\sigma \left(z^i_\sigma - x^i_\sigma\right). \tag{SP}$$

Subjective prices exist under the maintained hypotheses on preferences at interior consumption plans. Indeed, for every individual i in $\mathcal{J}$,

$$\left(p^i_\sigma\right)_{\sigma \in \mathcal{S}} = \left(\pi_\sigma \partial u^i\left(x^i_\sigma\right)\right)_{\sigma \in \mathcal{S}}.$$

2.7. **Stationarity.** In part of the analysis, we limit attention to stationary economies, rendering this restriction explicit when it occurs. An economy is stationary if uncertainty can be represented as a Markov process over a finite state space and preferences are measurable with respect to this state space. Formally, for some *finite* state space, S,

$$\mathcal{S} = \bigcup_{t \in \mathcal{T}} S^t,$$

where S^t denotes the set of histories of length t in $\mathcal{T}$. (The initial history ϕ in $\mathcal{S}$ is given by some state s_0 in S, that is, $S^0 = \{s_0\}$.) This induces an obvious finite partition $\left(\mathcal{S}^s\right)_{s \in S}$ of $\mathcal{S}$, given by the identification of every $\sigma = (s_0, s_1, \ldots, s_t)$ in $\mathcal{S}$ with the last state s_t in S appearing in the given history. Stationarity of the economy requires that, for every individual i in $\mathcal{J}$, the map

$$\sigma \mapsto \left(\frac{\pi^i_\tau}{\pi^i_\sigma}\right)_{\tau \in \sigma_+}$$

be measurable with respect to the finite partition $\left(\mathcal{S}^s\right)_{s \in S}$ of $\mathcal{S}$. Finally, in a stationary economy, an allocation x in X is stationary if it is measurable with respect to the finite partition $\left(\mathcal{S}^s\right)_{s \in S}$ of $\mathcal{S}$.

3. Inefficiency

An allocation x in X is *Pareto dominated* by an alternative allocation z in X if, for every individual i in $\mathcal{J}$,

$$U^i\left(z^i\right) - U^i\left(x^i\right) \geq 0$$

and, for some individual i in $\mathcal{J}$,

$$U^i\left(z^i\right) - U^i\left(x^i\right) > 0.$$

To introduce a general notion of constrained inefficiency, we allow for participation constraints at arbitrarily given reservation utilities. By varying reservation utilities, this serves to capture different notions of constrained inefficiency. We then restrict attention to reservation utilities induced by planned consumptions (or, simply, long-run inefficiency) and show that this is in fact the only relevant specification when debt constraints are consistent with reservation utilities.

Given reservation utilities ν in V, the vector space $\mathbb{R}^{\mathcal{S}\times\mathcal{J}}$, we define the set $X^{\text{PC}}(\nu)$ of all allocations z in X satisfying, for every individual i in $\mathcal{J}$, at every date-event σ in $\mathcal{S}$,

$$U_\sigma^i\left(z^i\right)-\nu_\sigma^i\geq 0. \tag{PC}$$

This is the set of allocations z in X fulfilling a sort of participation constraint when reservation utilities are given at values ν in V. By progressive specification, given an allocation e in X, we define the set $X^{\text{PC}}(e)$ of all allocations z in X satisfying, for every individual i in $\mathcal{J}$, at every date-event σ in $\mathcal{S}$,

$$U_\sigma^i\left(z^i\right)-U_\sigma^i\left(e^i\right)\geq 0.$$

This is the set of allocations z in X fulfilling a sort of participation constraint when reservation utilities are induced by allocation e in X, that is, allocations producing higher utility for all individuals beginning from any date-event with respect to the reference allocation.

An allocation x in X is *constrained inefficient at reservation utilities ν in V* if it is Pareto dominated by an allocation z in $X^{\text{PC}}(\nu)$ satisfying

$$\sum_{i\in\mathcal{J}}z^i\leq\sum_{i\in\mathcal{J}}x^i. \tag{CF}$$

By progressive specification, an allocation x in X is *constrained inefficient at allocation e in X* if it is constrained inefficient at reservation utilities induced by allocation e in X. Finally, an allocation x is X is simply *constrained inefficient* if it is constrained inefficient at allocation x in X.

The introduced notion of constrained inefficiency is strengthened in part of the analysis. Strong inefficiency occurs when, along some subtree of the economy, a welfare improving redistribution, satisfying participation constraints, is feasible even though a constant (however small) share of the aggregate endowment is destroyed. This redistribution, in addition, leaves consumptions unaltered in the remaining part of the economy. Formally, an allocation x in X is *strongly constrained inefficient at reservation utilities ν in V* if, for some non-empty subset $\mathcal{F}$ of $\mathcal{S}$ such that

$$\sigma\notin\mathcal{F}\text{ if and only if }\mathcal{F}\cap\mathcal{S}(\sigma)=\emptyset, \tag{T}$$

it is Pareto dominated by an allocation z in $X^{\text{PC}}(\nu)$ satisfying, for some $\epsilon>0$,

$$\epsilon u_{\mathcal{F}}+\sum_{i\in\mathcal{J}}z^i\leq\sum_{i\in\mathcal{J}}x^i \tag{SF-1}$$

and

$$\epsilon\sum_{i\in\mathcal{J}}\left|z^i-x^i\right|\leq u_{\mathcal{F}}, \tag{SF-2}$$

where $u_{\mathcal{F}}$ denotes the unit of the vector space

$$L_{\mathcal{F}}=\left\{v\in L:v_\sigma=0\text{ for every }\sigma\in(\mathcal{S}/\mathcal{F})\right\}.$$

Again, by progressive specification, an allocation x in X is *strongly constrained inefficient at allocation e in X* if it is constrained inefficient at reservation utilities induced by allocation e in X. Finally, an allocation x is X is simply *strongly constrained inefficient* if it is constrained inefficient at allocation x in X.

Strong inefficiency is meant to capture robust welfare losses occurring at equilibrium. In a tradition of general equilibrium (Debreu [13]), a measure of inefficiency is given by the *coefficient of resource utilization*, that is, by the largest share of the aggregate endowment whose destruction is consistent with a feasible welfare improving redistribution. Strong inefficiency occurs when this measure of inefficiency is positive. Over an infinite horizon, however, inefficiency might persist even though the mentioned measure of inefficiency vanishes. Though we do not explore this matter in depth, a non-strong inefficiency corresponds to the circumstance of benefits from the redistribution vanishing over time and, typically, of allocation approaching a constrained optimum at infinity. Finally, it is worth remarking that strong and canonical constrained inefficiency coincide if attention is limited to stationary allocations (§4).

Our main interest is for constrained inefficiency at planned consumptions. Constrained inefficiency at initial endowments is introduced for a comparison with Kehoe and Levine [17] and Alvarez and Jermann [5]. Constrained inefficiency at particular reservation utilities serves to encompass the formulation of Hellwig and Lorenzoni [16]. The first is suitable for a general characterization at any specification of debt constraints. The other two require a consistent specification of debt constraints. As far as our characterization is concerned, we clarify the exact differences and analogies in depth later on (§7).

4. Stationarity

We here show that, under the hypothesis of stationarity, every constrained inefficient allocation, among stationary allocations, is also strongly constrained inefficient. This might be regarded as a digression to ascertain the loss in generality induced by the strong form rather than the canonical form of inefficiency. Stationarity, indeed, ensures the existence of uniformly positive benefits from the welfare improving redistribution at constrained inefficient allocations.

Proposition 1 (Strongly *versus* canonical constrained inefficiency)**.** *In a stationary economy, a stationary interior allocation x in X is Pareto dominated by a stationary allocation z in $X^{\mathrm{PC}}(x)$, satisfying $\sum_{i \in \mathcal{J}} z^i \leq \sum_{i \in \mathcal{J}} x^i$, only if it is* strongly *constrained inefficient.*

The underlying logic can be illustrated as follows. At stationary allocations, inefficiency entails the comparison of *finitely many variations* in utility across date-events. In addition, by strict convexity of preferences, at no loss of generality, if the redistribution leaves utility unaltered at some date-event, then it also leaves consumptions unaltered at all succeeding date-events. Hence, a slight contraction of consumptions, at all date-events at which the redistribution occurs, preserves the strict increase in utility. This leaves uniformly positive quantities of resources undistributed at all date-events along some subtree of the economy.

5. Equilibrium

Individuals participate into financial markets, represented as a complete spectrum of elementary Arrow securities. However, their holdings of securities are restricted by quantitative limits. The nature of such debt, or solvency, constraints is irrelevant for the purpose of our analysis, insofar as consumption and financial plans of individuals do not bear any direct effect on debt constraints. In particular, the construction is consistent with that of Alvarez and Jermann [5] and of Hellwig and Lorenzoni [16], as well as with that of Bewley [7] (except for the fact that the formulation of Bewley [7] requires a single risk-less security, instead of a full set of elementary Arrow securities, available at every date-event).

At every date-event, simple Arrow securities are traded subject to solvency, or debt constraints. A price p is an element of

$$P = \left\{ p \in \mathbb{R}^{\mathcal{S}} : p_\sigma > 0 \text{ for every } \sigma \in \mathcal{S} \right\}.$$

Prices are expressed in present values and are comparable with Arrow-Debreu prices, or contingent prices, or state prices. Relevantly, prices need not assign finite values to (bounded) consumption plans over the entire infinite horizon. Thus, the duality between price and commodity spaces might fail, as in economies of overlapping generations.

Debt constraints are quantitative limits to liabilities held by individuals at non-initial date-events. For an individual i in $\mathcal{J}$, debt constraints f^i are an element of

$$F^i = \left\{ f^i \in \mathbb{R}^{\mathcal{S}} : f^i_\sigma \geq 0 \text{ for every non-initial } \sigma \in \mathcal{S} \right\}.$$

Across individuals, debt constraints f are elements of F. Notice that, as debt constraints are positive at non-initial date-events, saving is unrestricted, though borrowing might be inhibited by debt limits. In addition, to the only purpose of simplifying notation, the initial value of debt constraints serves to represent initial claims, or liabilities, held by individuals.

At price p in P, for an initial endowment of commodities e^i in X^i and debt constraints f^i in F^i, the budget set of individual i in $\mathcal{J}$ is given by

$$B^i_p\left(e^i, f^i\right) = \left\{ x^i \in X^i : \sum_{\tau \in \sigma_+} p_\tau v^i_\tau + p_\sigma \left(x^i_\sigma - e^i_\sigma\right) \leq p_\sigma v^i_\sigma \text{ for some } v^i \in V^i\left(f^i\right) \right\},$$

where

$$V^i\left(f^i\right) = \left\{ v^i \in \mathbb{R}^{\mathcal{S}} : v^i + f^i \geq 0, \text{ with } v^i_\phi + f^i_\phi = 0 \right\}.$$

The set $V^i\left(f^i\right)$ represents allowed financial plans. These are restricted by limits to debt and by given initial claims, or liabilities, both captured by f^i in F^i.

Debt constraints reflect solvency requirements. Under perfect financial markets, solvency is guaranteed whenever debt constraints do not exceed the present value of future endowment. However, when debt might not be honored, debt constraints serve to prevent incentives to default. Alvarez and Jermann [5] assume that, when default occurs, an individual is excluded from financial markets. Hellwig and Lorenzoni [16], instead, postulate that individuals are prohibited to borrow, though they might participate into financial markets for lending. Bewley [7] simply excludes borrowing and introduces positive outside money. Though the specific nature of debt constraints varies across all such instances, solvency requirements are specified as quantitative limits to the amount of liabilities held by individuals, so that they are all consistent with our representation of budget sets.

We are only concerned with prices that can be observed at equilibrium for *some* debt constraints and *some* initial endowments of commodities. Thus, the only relevant feature of equilibrium is optimality of consumption plans for individuals (that is, a sort of price support). A preliminary observation shows that it suffices to restrict attention to consumption plans that are optimal, for some debt constraints, when they are distributed to individuals as initial endowments. The logic is straightforward. If a consumption plan is optimal, it is sustained by some financial plan that satisfies some debt constraints. Thus, any *net* variation of this financial plan, consistent with given debt constraints, cannot yield higher utility. It follows that the consumption plan remains optimal when it corresponds to the initial endowment and debt constraints are given as the sum of initial debt constraints and the optimal financial plan. Clearly, in this transformation, saving and lending are to be interpreted as net positions, corresponding to variations with respect to the initial financial plan. For instance, if initial debt constraints prohibit borrowing (as in Bewley [7]), a negative net position, when the consumption plan is given as initial endowment, corresponds to a reduction of savings.

Proposition 2 (Price support)**.** *Given a price p in P, for every $\left(e^i, g^i\right)$ in $X^i \times F^i$, a consumption plan x^i in X^i is U^i-optimal in the budget set $B^i_p\left(e^i, g^i\right)$ only if, for some debt constraints f^i in F^i, it is U^i-optimal in the budget set $B^i_p\left(x^i, f^i\right)$.*

An allocation x in X is supported by price p in P at debt-constraints f in F if, for every individual i in $\mathcal{J}$, the consumption plan x^i in X^i is U^i-optimal in the budget constrain $B_p\left(x^i, f^i\right)$. An allocation x in X is *supported* (respectively, *non-trivially supported*) by price p in P if it is supported by price p in P at some debt-constraints f in F (respectively, at some debt constraints f in F satisfying, at every non-initial date-event σ in $\mathcal{S}$, $\sum_{i\in\mathcal{J}} f^i_\sigma > 0$). Non-trivial support requires that, at every date-event, some individual is allowed to borrow (*i.e.*, to reduce savings), so ruling out fundamentally autarchic equilibria.

Price support admits a first-order characterization based on elementary arbitrage arguments, as in Alvarez and Jermann [5]. First, for every individual, the subjective evaluation of transfers at succeeding date-events cannot exceed their market evaluation (FOC-1). Second, whenever an individual is allowed to borrow against income at some succeeding date-event, subjective and market evaluations need coincide (FOC-2). These necessary conditions are also sufficient for optimality, provided that boundedness of debt constraints ensures transversality.

Proposition 3 (First-order characterization)**.** *An interior allocation x in X is supported by price p in P at debt-constraints f in F only if, at every date-event σ in $\mathcal{S}$,*

$$\text{(FOC-1)} \qquad \bigvee_{i\in\mathcal{J}} \left(\frac{p^i_\tau}{p^i_\sigma}\right)_{\tau\in\sigma_+} \leq \left(\frac{p_\tau}{p_\sigma}\right)_{\tau\in\sigma_+}$$

and

$$\text{(FOC-2)} \qquad \sum_{\tau\in\sigma_+} \left(\frac{p^i_\tau}{p^i_\sigma}\right) f^i_\tau = \sum_{\tau\in\sigma_+} \left(\frac{p_\tau}{p_\sigma}\right) f^i_\tau,$$

where, for every individual i in $\mathcal{J}$, p^i in P^i is the subjective price at interior consumption plan x^i in X^i. Furthermore, an interior allocation x in X is supported by price p in P at bounded *debt-constraints f in F if, at the initial date-event ϕ in $\mathcal{S}$, $f_\phi = 0$ and, at every date-event σ in $\mathcal{S}$, conditions* (FOC-1)-(FOC-2) *are satisfied.*

6. Characterization

We here provide an equivalent characterization of constrained inefficiency, at planned consumptions, in terms of supporting prices. In particular, we show that prices reveal this sort of inefficiency independently of the nature of debt constraints. This characterization exploits a Modified Cass Criterion, exactly as in economies of overlapping generations. The Modified Cass Criterion is a variation of the original criterion proposed by Cass [11] for capital theory and was initially introduced by Bloise and Calciano [9] for stochastic overlapping generations economies.

A price p in P satisfies the Modified Cass Criterion if there exists a non-null positive element v of L satisfying, for some $1 > \rho > 0$, at every σ in $\mathcal{S}$,

$$\rho \sum_{\tau \in \sigma_+} p_\tau v_\tau \geq p_\sigma v_\sigma.$$

This criterion admits equivalent specifications in terms of weighted sum of the reciprocals of prices and of dominant root (and, to some extent, spectral radius) of a suitably defined linear operator (see Proposition 1 and Remarks 1-2 in Bloise and Calciano [9]), the latter being suitable for direct applications in empirical studies. The parameter $(\rho - 1)$ represents an appropriate estimation of (an upper bound on) the implicit average real rate of interest in the long-run.

Prices fulfilling the Modified Cass Criterion might be regarded as involving *low interest rates*. Prices exhibit *high interest rates*, according to the terminology borne out by Alvarez and Jermann [5], when they are summable, that is,

$$\sum_{\sigma \in \mathcal{S}} p_\sigma \text{ is finite.}$$

Clearly, high interest rates are inconsistent with the Modified Cass Criterion. Finally, prices involve *neither high nor low interest rates* when they neither satisfy the Modified Cass Criterion nor are summable. The latter circumstance reveals a null interest rate in the long period and corresponds to a *golden rule* in the terminology for overlapping generations economies. High interest rates, in turn, guarantee a finite pricing of all intertemporal consumption profiles, so preserving the duality between commodity and price spaces. As our characterization of inefficiency exploits low interest rates, prices are consistent with an efficient allocation of resources even when not involving high interest rates and, hence, an infinite value of the aggregate endowment.

In the formulation of Hellwig and Lorenzoni [16], when repudiating their debt, individuals are not excluded by participation into financial markets, though they are not allowed to hold liabilities anymore. Debt constraints are determined so as to prevent individuals from default and to sustain self-enforcing private debt. The characterization of Hellwig and Lorenzoni [16] shows that debt is self-enforcing if and only if debt constraints allow for exact roll-over, that is, in our notation, at every non-initial date-event σ in $\mathcal{S}$,

$$\sum_{\tau \in \sigma_+} p_\tau f_\tau = p_\sigma f_\sigma.$$

When debt constraints are bounded, exact roll-over implies that prices never involve high interest rates. Our Modified Cass Criterion is of particular relevance in this situation.

We begin with proving sufficiency of the Modified Cass Criterion.

Proposition 4 (Sufficiency)**.** *An interior allocation x in X, with* non-trivially *supporting price p in P, is constrained inefficient if there exists a non-null positive element v of L satisfying, for some $1 > \rho > 0$, at every σ in $\mathcal{S}$,*

$$\rho \sum_{\tau \in \sigma_+} p_\tau v_\tau \geq p_\sigma v_\sigma.$$

The logic underlying welfare improvement is extremely simple. For an elementary illustration, suppose that there is no uncertainty (that is, $\mathcal{S}$ can be identified with $\mathcal{T}$). By hypothesis, all consumption plans are interior and, at every date-event, at least one individual is unconstrained (that is, has a subjective evaluation of transfers to the following period coinciding with the market evaluation). Thus, at every period t in $\mathcal{T}$, for some unconstrained individual i in $\mathcal{J}$, the modification of consumptions, described by

$$\left(x_t^i, x_{t+1}^i\right) \mapsto \left(x_t^i - v_t, x_{t+1}^i + v_{t+1}\right),$$

induces a first-order effect on welfare that can be estimated as

$$-p_t^i v_t + p_{t+1}^i v_{t+1} = \left(\frac{p_t^i}{p_t}\right)\left(-p_t v_t + p_{t+1} v_{t+1}\right) \geq \left(\frac{1-\rho}{\rho}\right) p_t^i v_t.$$

This estimate exploits the fact that, for an unconstrained individual, subjective and market evaluations coincide. By iterating this sort of transfers across periods of trade, the final redistribution yields a positive first-order effect on utilities beginning from every period. As second-order effects are uniformly bounded, smoothness suffices to prove a welfare improvement beginning from all date-events.

We now prove necessity of the Modified Cass Criterion. This requires a strengthening of inefficiency to capture non-vanishing benefits from the redistribution across periods of trade. As mentioned earlier, this sort of strong inefficiency corresponds to inefficiency of positive measure according to Debreu's [13] coefficient of resource utilization.

Proposition 5 (Necessity)**.** *An interior allocation x in X, with supporting price p in P, is constrained inefficient only if there exists a non-null positive element v of L satisfying, at every σ in $\mathcal{S}$,*

$$\sum_{\tau \in \sigma_+} p_\tau v_\tau \geq p_\sigma v_\sigma.$$

In addition, it is strongly *constrained inefficient only if there exists a non-null positive element v of L satisfying, for some $1 > \rho > 0$, at every σ in $\mathcal{S}$,*

$$\rho \sum_{\tau \in \sigma_+} p_\tau v_\tau \geq p_\sigma v_\sigma.$$

Necessity is also straightforwardly explained. For every individual i in $\mathcal{S}$, at every date-event σ in $\mathcal{S}$,

$$\sum_{\tau \in \sigma_+} p_\tau^i v_\tau^i + p_\sigma^i \left(z_\sigma^i - x_\sigma^i\right) = p_\sigma^i v_\sigma^i.$$

Here, v^i in L represents the subjectively evaluated (first-order) benefit, in terms of current consumption, from the redistribution. This benefit needs be positive at all date-events. Thus, exploiting the fact that subjective evaluation cannot exceed market evaluation (FOC-1)-(FOC-2), at every date-event σ in $\mathcal{S}$,

$$\frac{1}{p_\sigma}\sum_{\tau\in\sigma_+} p_\tau v_\tau^i + \left(z_\sigma^i - x_\sigma^i\right) \geq v_\sigma^i.$$

Only market prices appear in this inequality. Aggregating across individuals,

$$\frac{1}{p_\sigma}\sum_{\tau\in\sigma_+} p_\tau \sum_{i\in\mathcal{J}} v_\tau^i + \sum_{i\in\mathcal{J}}\left(z_\sigma^i - x_\sigma^i\right) \geq \sum_{i\in\mathcal{J}} v_\sigma^i.$$

Feasibility proves the claim, as the aggregate subjectively evaluated benefit $\sum_{i\in\mathcal{J}} v^i$ in L satisfies

$$\sum_{\tau\in\sigma_+} p_\tau \sum_{i\in\mathcal{J}} v_\tau^i \geq p_\sigma \sum_{i\in\mathcal{J}} v_\sigma^i.$$

Finally, the strong form of inefficiency allows for a small uniform contraction preserving increasing subjectively evaluated gains from the redistribution.

7. Consistent Debt Constraints

We here verify to which extent our characterization is preserved under the notion of constrained inefficiency at given reservation utilities, rather than at planned consumptions. This allows for a direct comparison with the characterization provided by Alvarez and Jermann [5]. In addition, it provides insights into constrained inefficiency at equilibrium with self-enforcing debt as in Hellwig and Lorenzoni [16].

Sufficiency is obviously unaltered. If planned consumptions are individually rational at some given reservation utilities, any welfare improving reallocation guaranteeing sequential participation at planned consumptions is *a fortiori* a welfare improving reallocation guaranteeing sequential participation at the given reservation utilities.

Proposition 6 (Sufficiency with consistent debt constraints)**.** *Given reservation utilities ν in V, an interior allocation x in $X^{\text{PC}}(\nu)$, with* non-trivially *supporting price p in P, is constrained inefficient at reservation utilities ν in V if there exists a non-null positive element v of L satisfying, for some $1 > \rho > 0$, at every σ in $\mathcal{S}$,*

$$\rho \sum_{\tau\in\sigma_+} p_\tau v_\tau \geq p_\sigma v_\sigma.$$

As far as necessity is concerned, we preliminarily add restrictions on debt constraints consistent with Alvarez and Jermann's [5] and Hellwig and Lorenzoni's [16] formulations. Given reservation utilities ν in V, an allocation x in $X^{\text{PC}}(\nu)$ is supported by price p in P *at debt constraints consistent with reservation utilities ν in V* if it is supported by price p in P at debt constraints f in F satisfying, for every individual i in $\mathcal{J}$, at every non-initial date-event σ in $\mathcal{S}$,

$$U_\sigma^i\left(x^i\right) - \nu_\sigma^i > 0 \text{ only if } f_\sigma^i > 0. \tag{DC}$$

The underlying logic of this notion is that, whenever subjective welfare exceeds reservation utility at some date-event, debt constrains allow for borrowing at that date-event, that is, for (locally) unrestricted participation into financial markets.

Proposition 7 (Necessity with consistent debt constraints)**.** *Given reservation utilities ν in V, an interior allocation x in $X^{\mathrm{PC}}(\nu)$, with supporting price p in P at debt constraints consistent with reservation utilities ν in V, is constrained inefficient at reservation utilities ν in V only if there exists a non-null positive element v of L satisfying, at every σ in $\mathcal{S}$,*

$$\sum_{\tau \in \sigma_+} p_\tau v_\tau \geq p_\sigma v_\sigma.$$

In addition, it is strongly *constrained inefficient at reservation utilities ν in V only if there exists a non-null positive element v of L satisfying, for some $1 > \rho > 0$, at every σ in $\mathcal{S}$,*

$$\rho \sum_{\tau \in \sigma_+} p_\tau v_\tau \geq p_\sigma v_\sigma.$$

The proof of this claim requires a minor amendment of the previous argument for necessity (proposition 5). For an individual i in $\mathcal{J}$, the subjectively evaluated benefit from the redistribution v^i in L need not be positive at all date-events, though it is positive at the initial date-event. (Indeed, at some non-initial date-event, subjective welfare might fall below utility at planned consumptions.) However, notice that, when an individual is constrained in transferring resources at a date-event, consistent debt constraints ensure that the individual will positively benefit, with respect to planned consumptions, from the redistribution at that date-event. Hence, for every individual i in $\mathcal{J}$, at every date-event σ in $\mathcal{S}$,

$$\frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau v_\tau^i + \left(z_\sigma^i - x_\sigma^i\right) \geq v_\sigma^i.$$

The argument then unfolds as in the proof of proposition 5, using only the non-null positive part of aggregate subjectively evaluated benefit $\sum_{i \in \mathcal{J}} v^i$ in L.

Loosely interpreted, our complete characterization proves that constrained inefficiency at initial endowments (that is, constrained inefficiency as defined by Kehoe and Levine [17] and Alvarez and Jermann [5]) coincides with *low interest rates.* Alvarez and Jermann [5] show, on the one side, that every equilibrium allocation involving high interest rates is constrained efficient (Corollary 4.7) and, on the other side, that every non-autarchic constrained efficient allocation involves high interest rates (Proposition 4.10). Therefore, according to Alvarez and Jermann [5], *high interest rates* fully characterize non-autarchic constrained efficiency. What about *neither high nor low interest rates*, that is, a null interest rate over the long period?

In appendix A, we provide an example of a non-autarchic equilibrium with not-too-tight debt constraints that it is constrained efficient at the initial allocation and it involves a constant null interest rate. This example, though it requires non-stationary initial endowments, shows that a null interest rate over the long period can be sustained at non-autarchic equilibrium with not-too-tight debt constraints. In turn, non-stationary endowments might be of interest for applications to the sustainability of sovereign debt, when some countries face a decline, or a deindustrialization, and some other countries an expansion, or an industrialization. Consistently, our characterization is tight.

In appendix B, we also complement Kehoe and Levine's [17] and Alvarez and Jermann's [5] Second Welfare Theorem in order to prove that, when both consumptions and endowments are stationary, a non-autarchic constrained optimum requires high interest rates. As a conclusion, limiting attention to non-autarchic *stationary* consumptions, constrained efficiency at initial *stationary* endowments is fully characterized by *high interest rates*.

8. Conclusion

We have shown that the Modified Cass Criterion fully reveals constrained inefficiency at equilibrium with any sort of debt constraints, when constrained inefficiency corresponds to the occurrence of a feasible welfare improvement beginning from every contingency. When debt constraints are specified consistently with some reservation utilities, an analogous characterization emerges, when constrained inefficiency coincides with a feasible *ex-ante* welfare improvement subject to participation constrains at the given reservation utilities. This shows that the nature of constrained inefficiency at equilibrium with consistent debt constraints is basically that captured by our notion of constrained inefficiency, that is, a recursively feasible welfare improvement.

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Appendix A. Example

In this appendix, we provide an example of a constrained efficient allocation, according to Alvarez and Jermann [5], violating the hypothesis of high interest rates. In particular, a null interest rate sustains a stationary allocation as non-autarchic equilibrium at not-too-tight debt constraints. Initial endowments are non-stationary and are constructed so as to approach the equilibrium stationary allocation in the long period. Non-stationarity of either endowments or consumptions is necessary for a non-autarchic constrained optimum not to involve high interest rates, as shown in appendix B.

Before presenting the example, we shall produce necessary conditions for constrained inefficiency. To simplify, we shall assume that there is no uncertainty, that is, $\mathcal{S}$ can be identified with $\mathcal{T}$; also, that there is a common discount factor, $1 > \delta > 0$, and that the common per-period utility function $u : \mathbb{R}_+ \to \mathbb{R}$ is smooth on $\mathbb{R}_+$ (that is, to be precise, it can be extended as a twice continuously differentiable function on some open set containing $\mathbb{R}_+$); finally, that $u'(1) < \delta u'(0)$. Recall that, given an initial allocation e in X, an interior allocation x in $X^{\text{PC}}(e)$ is supported by price p in P at not-too-tight debt constraints with respect to initial allocation e in X if it is supported by price p in P such that, for every individual i in $\mathcal{J}$, at every t in $\mathcal{T}$,

$$\text{(FOC-1)} \qquad \frac{p_{t+1}}{p_t} \geq \frac{p^i_{t+1}}{p^i_t}$$

and

$$\text{(FOC-2)} \qquad \frac{p_{t+1}}{p_t} = \frac{p^i_{t+1}}{p^i_t} \text{ if } U^i_{t+1}\left(x^i\right) - U^i_{t+1}\left(e^i\right) > 0,$$

where p^i in P^i is the subjective price at interior consumption plan x^i in X^i.

Claim 1 (Constrained inefficiency)**.** *Given an initial allocation e in X, an interior allocation x in $X^{\text{PC}}(e)$, with supporting price p in P at not-too-tight debt constraints with respect to initial allocation e in X, is Pareto dominated by an allocation z in $X^{\text{PC}}(e)$, satisfying $\sum_{i\in\mathcal{J}} z^i \leq \sum_{i\in\mathcal{J}} x^i$, only if there exists a strictly positive* *element v of L satisfying, for some sufficiently small $\epsilon > 0$, at every t in $\mathcal{T}$,*

$$\frac{p_{t+1}}{p_t} v_{t+1} \geq v_t + \epsilon \sum_{i \in \mathcal{J}} \left(z_t^i - x_t^i\right)^2$$

and

$$\sum_{s \in \mathcal{T}} \delta^s \sum_{i \in \mathcal{J}} \left|z_{t+s}^i - x_{t+s}^i\right| \geq \epsilon v_t.$$

Proof of claim 1. Preliminarily observe that, for consumptions varying in a compact interval of $\mathbb{R}_+$, there exists a sufficiently small $\epsilon > 0$ satisfying

$$u(c') - u(c) \leq u'(c)(c' - c) - \epsilon u'(c)(c' - c)^2.$$

This shows a sort of quadratic concavity of intertemporal utility.

For every individual i in $\mathcal{J}$, at every t in $\mathcal{T}$, define

$$v_t^i = \frac{1}{p_t^i} \sum_{s \in \mathcal{T}} p_{t+s}^i \left(z_{t+s}^i - x_{t+s}^i\right) - \epsilon \frac{1}{p_t^i} \sum_{s \in \mathcal{T}} p_{t+s}^i \left(z_{t+s}^i - x_{t+s}^i\right)^2.$$

Notice that, for every individual i in $\mathcal{J}$, v^i is an element of L. Define $v = \sum_{i \in \mathcal{J}} v^i$, an element itself of L, and observe that, by Pareto dominance and quadratic concavity, $v_0 = \sum_{i \in \mathcal{J}} v_0^i > 0$. In addition, at every t in $\mathcal{T}$,

$$\frac{1}{\epsilon} \sum_{i \in \mathcal{J}} \sum_{s \in \mathcal{T}} \delta^s \left|z_{t+s}^i - x_{t+s}^i\right| \geq \sum_{i \in \mathcal{J}} \frac{1}{p_t^i} \sum_{s \in \mathcal{T}} p_{t+s}^i \left|z_{t+s}^i - x_{t+s}^i\right| \geq v_t,$$

where the first inequality, as $\epsilon > 0$ can be assumed to be arbitrarily small, follows from bounded derivatives of per-period utility $u : \mathbb{R}_+ \to \mathbb{R}$ over a compact interval of $\mathbb{R}_+$.

For every individual i in $\mathcal{J}$, at every t in $\mathcal{T}$,

$$\frac{p_{t+1}^i}{p_t^i} v_{t+1}^i + \left(z_t^i - x_t^i\right) \geq v_t^i + \epsilon \left(z_t^i - x_t^i\right)^2.$$

As debt constraints are not-too-tight,

$$\frac{p_{t+1}}{p_t} > \frac{p_{t+1}^i}{p_t^i} \text{ only if } U_{t+1}^i\left(x^i\right) - U_{t+1}^i\left(e^i\right) = 0.$$

Hence, as $U_{t+1}^i\left(z^i\right) - U_{t+1}^i\left(x^i\right) \geq 0$, $v_{t+1}^i \geq 0$. We consistently conclude that, for every individual i in $\mathcal{J}$, at every t in $\mathcal{T}$,

$$\frac{p_{t+1}}{p_t} v_{t+1}^i + \left(z_t^i - x_t^i\right) \geq v_t^i + \epsilon \left(z_t^i - x_t^i\right)^2.$$

Aggregating across individuals, by feasibility, this proves our claim. □

For the example, it suffices to consider only two individuals, $\mathcal{J} = \{e, o\}$, associated with even, e, and odd, o, periods of trade. Let $x_e > 0$ and $x_o > 0$ satisfy $x_e + x_o = 1$ and

$$u'(x_e) = \delta u'(x_o). \tag{ss}$$

Allocation x in X is given by

$$\begin{aligned} x^e &= (x_e, x_o, x_e, x_o, \ldots), \\ x^o &= (x_o, x_e, x_o, x_e, \ldots). \end{aligned}$$

At allocation x in X, the supporting price p in P is

$$(p_t)_{t\in\mathcal{T}} = (1,1,1,\ldots,1,\ldots),$$

whereas the subjective price p^i in P^i of individual i in $\mathcal{J}$ is given by

$$\left(p_t^i\right)_{t\in\mathcal{T}} = \left(\delta^t u'\left(x_t^i\right)\right)_{t\in\mathcal{T}}.$$

We need to construct initial endowments e in X which are consistent with price support at not-too-tight debt constraints.

Claim 2 (Not-too-tight debt constraints)**.** *There exists an initial allocation e in X, satisfying $\sum_{i\in\mathcal{J}} x^i = \sum_{i\in\mathcal{J}} e^i$, such that allocation x in $X^{\mathrm{PC}}(e)$ is supported by price p in P at not-too-tight debt constraints with respect to initial allocation e in X.*

Proof of claim 2. Consider the (local) difference equation

$$(*) \qquad h(\xi_t,\xi_{t+1}) = u(x_e) + \delta u(x_o) - u(x_e+\xi_t) - \delta u(x_o - \xi_{t+1}) = 0.$$

It is easy to verify that this difference equation admits a strictly positive solution $(\xi_t)_{t\in\mathcal{T}}$ in L satisfying $\lim_{t\in\mathcal{T}} \xi_t = 0$. (Indeed, observe that $\xi > 0$ implies $h(\xi,\xi) > 0$ and $h(\xi,0) < 0$, so that $h(\xi,\xi') = 0$ for some $\xi > \xi' > 0$ by the Intermediate Value Theorem.) Endowments e in X are given by

$$\begin{aligned} e^e &= (x_e+\xi_0, x_o-\xi_1, x_e+\xi_2, x_o-\xi_3,\ldots), \\ e^o &= (x_o-\xi_0, x_e+\xi_1, x_o-\xi_2, x_e+\xi_3,\ldots). \end{aligned}$$

In addition, because of restriction (*), at every t in $\{0,2,4,\ldots\}$,

$$U_t^e(x^e) = U_t^e(e^e)$$

and

$$U_t^o(x^o) \geq u(x_o) + \delta U_{t+1}^o(x^o) > u(x_o - \xi_t) + \delta U_{t+1}^o(e^o) \geq U_t^o(e^o);$$

at every t in $\{1,3,5,\ldots\}$,

$$U_t^o(x^o) = U_t^o(e^o)$$

and

$$U_t^e(x^e) \geq u(x_o) + \delta U_{t+1}^e(x^e) > u(x_o - \xi_t) + \delta U_{t+1}^e(e^e) \geq U_t^e(e^e).$$

This suffices to prove the claim. □

We now conclude that allocation x in X is a constrained optimum at initial allocation e in X.

Claim 3 (Constrained optimum)**.** *Given the constructed initial allocation e in X, allocation x in $X^{\mathrm{PC}}(e)$ is not Pareto dominated by an alternative allocation z in $X^{\mathrm{PC}}(e)$ satisfying $\sum_{i\in\mathcal{J}} z^i \leq \sum_{i\in\mathcal{J}} x^i$.*

Proof of claim 3. Supposing not, because of claim 2, we can apply the characterization of claim 1. Exploiting the stationarity of supporting price p in P, this characterization imposes the existence of a strictly positive element v of L satisfying, for some sufficiently small $\epsilon > 0$, at every t in $\mathcal{T}$,

$$(*) \qquad v_{t+1} \geq v_t + \epsilon \sum_{i\in\mathcal{J}} \left(z_t^i - x_t^i\right)^2$$

and

$$(**) \qquad \sum_{s\in\mathcal{T}} \delta^s \sum_{i\in\mathcal{J}} \left| z^i_{t+s} - x^i_{t+s} \right| \geq \epsilon v_t.$$

Clearly, the sequence $(v_t)_{t\in\mathcal{T}}$ in L converges, so that condition (*) yields

$$\lim_{t\in\mathcal{T}} v_{t+1} \geq v_0 + \epsilon \sum_{t\in\mathcal{T}} \sum_{i\in\mathcal{J}} \left(z^i_t - x^i_t \right)^2.$$

Therefore,

$$\lim_{t\in\mathcal{T}} \sum_{i\in\mathcal{J}} \left| z^i_t - x^i_t \right| = 0.$$

This is inconsistent with condition (**) as the sequence $(v_t)_{t\in\mathcal{T}}$ in L is (weakly) increasing. □

Summing up, we have provided an example of a constrained optimum, according to Alvarez and Jermann [5], which is not autarchic and does not involve high interest rates, as supporting prices exhibit a null interest rate. It is to be remarked that, *strictu sensu*, this is not a counter-example to Proposition 4.10 of Alvarez and Jermann [5], as they also assume stationary endowments, though, in the proof, stationarity of endowments seems not being exploited.

Appendix B. Second Welfare Theorem

We here provide a version of the Second Welfare Theorem as in Kehoe and Levine [17, Proposition 5]. The Second Welfare Theorem of Kehoe and Levine [17] is exploited by Alvarez and Jermann [5, Proposition 4.10] to prove necessity of high interest rates at non-autarchic constrained efficient allocations.

Given an initial allocation e in X, an allocation x in $X^{\text{PC}}(e)$ is an *abstract equilibrium with transfers at initial allocation e in X* if there exists a positive linear functional φ on L such that, given any allocation z in $X^{\text{PC}}(e)$, for every individual i in $\mathcal{J}$,

$$U^i\left(z^i\right) - U^i\left(x^i\right) > 0 \text{ implies } \varphi \cdot \left(z^i - x^i\right) > 0.$$

Claim 4 (Second Welfare Theorem under Stationarity)**.** *In a stationary economy, given a stationary allocation e in X, a stationary interior allocation x in $X^{\text{PC}}(e)$, satisfying*

$$\sum_{i\in\mathcal{J}} x^i - \sum_{i\in\mathcal{J}} e^i = 0$$

and

$$(\text{sw}) \qquad \sum_{i\in\mathcal{J}} U^i_\sigma\left(x^i\right) - \sum_{i\in\mathcal{J}} U^i_\sigma\left(e^i\right) > 0 \text{ at every } \sigma \in \mathcal{S},$$

is not constrained inefficient at initial allocation e in X only if it is an abstract equilibrium with transfers at initial allocation e in X.

Proof of claim 4. By the Separating Hyperplane Theorem (see Kehoe and Levine [17]), there exists a non-null positive linear functional φ on L such that, for every allocation z in $X^{\text{PC}}(e)$ that (weakly) Pareto dominates allocation x in X,

$$\sum_{i\in\mathcal{J}} \varphi \cdot \left(z^i - x^i\right) \geq 0.$$

Clearly, by positivity of the supporting linear functional, $\varphi \cdot u > 0$, where u is any interior positive element of L. We shall prove that the linear functional φ on L is strictly positive (that is, for every non-null positive element v of L, $\varphi \cdot v > 0$). By canonical arguments, this suffices to prove the claim.

Assuming not, then there exists $v > 0$ in L such that $\varphi \cdot v = 0$ and, for all but finitely many σ in $\mathcal{S}$, $v_\sigma = 0$. For any sufficiently small $1 > \lambda > 0$, consider the interior allocation z in X that is defined, for every individual i in $\mathcal{J}$, by

$$z^i = (1 - \lambda) x^i + \lambda e^i + v.$$

By strict monotonicity and continuity of preferences, allocation z in X strictly Pareto dominates allocation x in X, provided that $1 > \lambda > 0$ is sufficiently small. By strict monotonicity and strict convexity of preferences, allocation z lies in $X^{\mathrm{PC}}(e)$ and, in addition, for every individual i in $\mathcal{J}$, at every σ in $\mathcal{S}$,

$$(*) \qquad U_\sigma^i \left(z^i\right) - U_\sigma^i \left(e^i\right) = 0 \text{ implies } \left(z_\tau^i\right)_{\tau \in \mathcal{S}(\sigma)} = \left(e_\tau^i\right)_{\tau \in \mathcal{S}(\sigma)}.$$

Also, consider the collection $\left(\mathcal{F}^i\right)_{i \in \mathcal{J}}$ determined, for every individual i in $\mathcal{J}$, by $\mathcal{F}^i = \left\{\sigma \in \mathcal{S} : U_\sigma^i \left(z^i\right) - U_\sigma^i \left(e^i\right) > 0\right\}$. Notice that, by stationarity, provided that $1 > \lambda > 0$ is sufficiently small, it can be assumed that, for every individual i in $\mathcal{J}$, $\left\{\sigma \in \mathcal{S} : U_\sigma^i \left(x^i\right) - U_\sigma^i \left(e^i\right) > 0\right\} \subset \mathcal{F}^i$, so that, using condition (SW),

$$(**) \qquad \bigcup_{i \in \mathcal{J}} \mathcal{F}^i = \mathcal{S}.$$

Finally, observe that, as v in L vanishes at all but finitely many date-events σ in $\mathcal{S}$, for every individual i in $\mathcal{J}$, the map

$$\sigma \mapsto \left(z_\tau^i\right)_{\tau \in \mathcal{S}(\sigma)}$$

is measurable with respect to some finite partition of $\mathcal{S}$.

By the last observation and restriction (*), there exists $1 > \theta > 0$ such that the alternative interior allocation y in X, defined, for every individual i in $\mathcal{J}$, by

$$y^i = z^i - \theta \sum_{\sigma \in \mathcal{F}^i} x_\sigma^i,$$

lies in $X^{\mathrm{PC}}(e)$ and Pareto dominates allocation x in X. (Here, to simplify notation, we use the decomposition $\mathbb{R}^{\mathcal{S}} = \oplus_{\sigma \in \mathcal{S}} \mathbb{R}_\sigma$.) Hence, by separation,

$$(\#\mathcal{J})\, \varphi \cdot v - \theta \varphi \cdot \sum_{i \in \mathcal{J}} \sum_{\sigma \in \mathcal{F}^i} x_\sigma^i \geq \varphi \cdot \left(\sum_{i \in \mathcal{J}} y^i - \sum_{i \in \mathcal{J}} x^i\right) \geq 0,$$

that is,

$$0 \geq \left(\frac{\#\mathcal{J}}{\theta}\right) \varphi \cdot v \geq \varphi \cdot \sum_{i \in \mathcal{J}} \sum_{\sigma \in \mathcal{F}^i} x_\sigma^i.$$

Observing that allocation x in X is interior and that condition (**) holds, this is a contradiction, as $\varphi \cdot u > 0$ for every interior positive element u of L. $\square$

Appendix C. Proofs

Proof of proposition 1. The stationary allocation x in X is Pareto dominated by an alternative stationary allocation z in $X^{\text{PC}}(x)$ satisfying

$$\sum_{i \in \mathcal{J}} z^i \leq \sum_{i \in \mathcal{J}} x^i.$$

At no loss of generality, for every individual i in $\mathcal{J}$, at every σ in $\mathcal{S}$,

$$U^i_\sigma\left(z^i\right) - U^i_\sigma\left(x^i\right) = 0 \text{ implies } \left(z^i_\tau\right)_{\tau \in \mathcal{S}(\sigma)} = \left(x^i_\tau\right)_{\tau \in \mathcal{S}(\sigma)}.$$

(Indeed, if not, by strict convexity of preferences, one could use, for some sufficiently large $1 > \lambda > 0$, the alternative stationary allocation $\lambda(z - x) + x$ in X.) For an individual i in $\mathcal{S}$, let $\mathcal{F}^i$ be the set consisting of all date-events σ in $\mathcal{S}$ such that

$$U^i_\sigma\left(z^i\right) - U^i_\sigma\left(x^i\right) > 0.$$

For some $1 > \lambda > 0$, define an alternative allocation y in X by setting, for every individual i in $\mathcal{J}$,

$$y^i = z^i - \lambda \sum_{\sigma \in \mathcal{F}^i} z^i_\sigma.$$

(For notational convenience, we use the decomposition $\mathbb{R}^{\mathcal{S}} = \oplus_{\sigma \in \mathcal{S}} \mathbb{R}_\sigma$.) By stationarity of preferences, there exists a sufficiently small $1 > \lambda > 0$ preserving welfare improvement. (This is so because stationarity requires to satisfy welfare improvement for finitely many continuous utility functions.) By interiority of allocation x in X, strong constrained inefficiency obtains at the subtree

$$\mathcal{F} = \bigcup_{i \in \mathcal{J}} \mathcal{F}^i.$$

This proves the claim. □

Proof of proposition 2. As consumption plan x^i in X^i is optimal in the budget set $B^i_p\left(e^i, g^i\right)$, for some financial plan v^i in $V^i\left(g^i\right)$, at every date-event σ in $\mathcal{S}$,

$$\sum_{\tau \in \sigma_+} p_\tau v^i_\tau + p_\sigma\left(x^i_\sigma - e^i_\sigma\right) = p_\sigma v^i_\sigma. \tag{*}$$

Consider debt constraints $f^i = v^i + g^i$ in F^i, which are positive as v^i is in $V^i\left(g^i\right)$. Suppose that consumption plan z^i in X^i lies in the budget set $B^i_p\left(x^i, f^i\right)$. It follows that, for some financial plan w^i in $V^i\left(f^i\right)$, at every date-event σ in $\mathcal{S}$,

$$\sum_{\tau \in \sigma_+} p_\tau w^i_\tau + p_\sigma\left(z^i_\sigma - x^i_\sigma\right) \leq p_\sigma w^i_\sigma.$$

Hence, at every date-event σ in $\mathcal{S}$,

$$-\sum_{\tau \in \sigma_+} p_\tau v^i_\tau + \sum_{\tau \in \sigma_+} p_\tau\left(w^i_\tau + v^i_\tau\right) + p_\sigma\left(z^i_\sigma - x^i_\sigma\right) \leq p_\sigma\left(w^i_\sigma + v^i_\sigma\right) - p_\sigma v^i_\sigma.$$

That is, using condition (*),

$$\sum_{\tau \in \sigma_+} p_\tau\left(w^i_\tau + v^i_\tau\right) + p_\sigma\left(z^i_\sigma - e^i_\sigma\right) \leq p_\sigma\left(w^i_\sigma + v^i_\sigma\right).$$

In addition, as w^i lies in $V^i\left(f^i\right)$, financial plan $w^i + v^i$ is an element of $V^i\left(g^i\right)$. It follows that consumption plan z^i in X^i belongs to the budget set $B^i_p\left(e^i, g^i\right)$, so proving the claim. □

Proof of proposition 3. Necessity of this first-order characterization is established by Alvarez and Jermann [5]. To prove sufficiency, for an individual i in $\mathcal{J}$, observe that consumption plan x^i lies in the budget set $B_p^i\left(x^i, f^i\right)$ and consider any consumption plan z^i in the budget set $B_p^i\left(x^i, f^i\right)$. It follows that, for some financial plan v^i in $V^i\left(f^i\right)$, at every date-event σ in $\mathcal{S}$,

$$-p_\sigma^i \sum_{\tau\in\sigma_+}\left(\frac{p_\tau}{p_\sigma}\right) f_\tau^i + p_\sigma^i \sum_{\tau\in\sigma_+}\left(\frac{p_\tau}{p_\sigma}\right)\left(v_\tau^i + f_\tau^i\right) + p_\sigma^i\left(z_\sigma^i - x_\sigma^i\right) \leq p_\sigma^i v_\sigma^i,$$

where p^i in P^i is the subjective price at consumption plan x^i in X^i. Using condition (FOC-1), along with the fact that v^i lies in $V^i\left(f^i\right)$, this yields

$$-p_\sigma^i \sum_{\tau\in\sigma_+}\left(\frac{p_\tau}{p_\sigma}\right) f_\tau^i + \sum_{\tau\in\sigma_+} p_\tau^i\left(v_\tau^i + f_\tau^i\right) + p_\sigma^i\left(z_\sigma^i - x_\sigma^i\right) \leq p_\sigma^i v_\sigma^i.$$

Using condition (FOC-2), this finally becomes

$$-\sum_{\tau\in\sigma_+} p_\tau^i f_\tau^i + \sum_{\tau\in\sigma_+} p_\tau^i\left(v_\tau^i + f_\tau^i\right) + p_\sigma^i\left(z_\sigma^i - x_\sigma^i\right) \leq p_\sigma^i\left(v_\sigma^i + f_\sigma^i\right) - p_\sigma^i f_\sigma^i.$$

Adding up, one obtains

$$-\sum_{\sigma\in\mathcal{S}_t}\sum_{\tau\in\sigma_+} p_\tau^i f_\tau^i + \sum_{\sigma\in\mathcal{S}^t} p_\sigma^i\left(z_\sigma^i - x_\sigma^i\right) \leq 0,$$

where, for every t in $\mathcal{T}$, $\mathcal{S}_t = \{\sigma\in\mathcal{S} : t(\sigma) = t\}$ and $\mathcal{S}^t = \{\sigma\in\mathcal{S} : t(\sigma) \leq t\}$. Observing that debt-constrains f in F are bounded and subjective price p^i in P^i defines an order-continuous linear functional on L,

$$\sum_{\sigma\in\mathcal{S}} p_\sigma^i\left(z_\sigma^i - x_\sigma^i\right) \leq 0.$$

This, because of (SP), suffices to prove the claim. □

Proof of proposition 4. At no loss of generality, as x in X is an interior allocation, it can be assumed that

$$\bigwedge_{i\in\mathcal{J}} x^i \geq v.$$

Consider a partition $\left(\mathcal{P}^i\right)_{i\in\mathcal{J}}$ of the set of *non-initial* date-events in $\mathcal{S}$ such that, for every non-initial date-event σ in $\mathcal{S}$, σ belongs to $\mathcal{P}^i$ only if $f_\sigma^i > 0$. This construction is consistent as price support is non-trivial. Also, for every individual i in $\mathcal{J}$, let $\mathcal{N}^i = \left\{\sigma\in\mathcal{S} : \sigma_+ \cap \mathcal{P}^i \neq \emptyset\right\}$. Finally, for every date-event σ in $\mathcal{S}$, define $\mathcal{P}^i(\sigma) = \mathcal{P}^i \cap \mathcal{S}(\sigma)$ and $\mathcal{N}^i(\sigma) = \mathcal{N}^i \cap \mathcal{S}(\sigma)$.

For every individual i in $\mathcal{J}$, define

$$z^i = x^i + \sum_{\sigma\in\mathcal{P}^i} v_\sigma - \sum_{\sigma\in\mathcal{N}^i}\left(\frac{\sum_{\tau\in\sigma_+\cap\mathcal{P}^i} p_\tau v_\tau}{\sum_{\tau\in\sigma_+} p_\tau v_\tau}\right) v_\sigma.$$

(For notational convenience, we use the decomposition $\mathbb{R}^{\mathcal{S}} = \oplus_{\sigma\in\mathcal{S}}\mathbb{R}_\sigma$.) For every individual i in $\mathcal{J}$, the underlying redistribution increases consumption at date-events in $\mathcal{P}^i$ and decreases consumption at date-events in $\mathcal{N}^i$. Clearly, z in X is a feasible allocation, that is, it satisfies (CF). Also, notice that, by construction, for every individual i in $\mathcal{J}$, at every date-event σ in $\mathcal{S}$,

$$\begin{aligned}
\sum_{\nu\in\mathcal{N}^i(\sigma)}\left(\frac{\sum_{\tau\in\nu_+\cap\mathcal{P}^i}p_\tau v_\tau}{\sum_{\tau\in\nu_+}p_\tau v_\tau}\right)p^i_\nu v_\nu &\leq \sum_{\nu\in\mathcal{N}^i(\sigma)}p^i_\nu\left(\frac{p_\nu v_\nu}{\sum_{\tau\in\nu_+}p_\tau v_\tau}\right)\frac{1}{p_\nu}\sum_{\tau\in\nu_+\cap\mathcal{P}^i}p_\tau v_\tau \\
&\leq \sum_{\nu\in\mathcal{N}^i(\sigma)}\left(\frac{p_\nu v_\nu}{\sum_{\tau\in\nu_+}p_\tau v_\tau}\right)\sum_{\tau\in\nu_+\cap\mathcal{P}^i}p^i_\tau v_\tau \\
&\leq \rho\sum_{\nu\in\mathcal{N}^i(\sigma)}\sum_{\tau\in\nu_+\cap\mathcal{P}^i}p^i_\tau v_\tau \\
&\leq \rho\sum_{\tau\in\mathcal{P}^i(\sigma)}p^i_\tau v_\tau.
\end{aligned}$$

The first inequality is a simple manipulation; the second inequality uses the fact that subjective and market evaluations coincide; the third inequality is a consequence of the Modified Cass Criterion; the last inequality uses the construction of subsets $\mathcal{P}^i$ and $\mathcal{N}^i$ of $\mathcal{S}$. Hence, for every individual i in $\mathcal{J}$, at every σ in $\mathcal{S}$,

$$(*)\qquad \sum_{\tau\in\mathcal{S}(\sigma)}p^i_\tau\left(z^i_\tau-x^i_\tau\right)\geq(1-\rho)\sum_{\tau\in\mathcal{P}^i(\sigma)}p^i_\tau v_\tau\geq(1-\rho)\sum_{\tau\in\mathcal{S}(\sigma)}p^i_\tau\left(z^i_\tau-x^i_\tau\right)^+.$$

Manipulating inequality (*), we obtain

$$\sum_{\tau\in\mathcal{S}(\sigma)}p^i_\tau\left(z^i_\tau-x^i_\tau\right)\geq\left(\frac{1-\rho}{\rho}\right)\sum_{\tau\in\mathcal{S}(\sigma)}p^i_\tau\left(z^i_\tau-x^i_\tau\right)^-\geq(1-\rho)\sum_{\tau\in\mathcal{S}(\sigma)}p^i_\tau\left(z^i_\tau-x^i_\tau\right)^-.$$

Hence, for every individual i in $\mathcal{J}$, at every σ in $\mathcal{S}$,

$$(**)\qquad \sum_{\tau\in\mathcal{S}(\sigma)}p^i_\tau\left(z^i_\tau-x^i_\tau\right)\geq\left(\frac{1-\rho}{2}\right)\sum_{\tau\in\mathcal{S}(\sigma)}p^i_\tau\left|z^i_\tau-x^i_\tau\right|.$$

Condition (**) guarantees a first-order positive welfare effect beginning from every date-event σ in $\mathcal{S}$. To obtain a welfare improvement, we show that higher order effects are uniformly bounded. As allocation x in X is interior, for a sufficiently small $\epsilon>0$, any allocation y in $B_\epsilon(x)$ is also interior, where

$$B_\epsilon(x)=\left\{y\in X:\sum_{i\in\mathcal{J}}\left\|y^i-x^i\right\|\leq\epsilon\right\}.$$

Notice that per-period utility $u^i:\mathbb{R}_+\to\mathbb{R}$ exhibits a bounded second-order term over any compact interval in $\mathbb{R}_{++}$. Thus, it can be assumed that there exists a sufficiently large $\mu>0$ satisfying, given any allocation y in $B_\epsilon(x)$, for every individual i in $\mathcal{J}$, at every σ in $\mathcal{S}$,

$$u^i\left(y^i_\sigma\right)-u^i\left(x^i_\sigma\right)\geq\partial u^i\left(x^i_\sigma\right)\left(y^i_\sigma-x^i_\sigma\right)-\left(\frac{\mu}{2}\right)\left|y^i_\sigma-x^i_\sigma\right|\partial u^i\left(x^i_\sigma\right)\left|y^i_\sigma-x^i_\sigma\right|.$$

Also, possibly contracting v in L, at no loss of generality,

$$\bigvee_{i\in\mathcal{J}}\left\|z^i-x^i\right\|\leq\|v\|\leq\epsilon\wedge\left(\frac{1-\rho}{\mu}\right).$$

Hence, for every individual i in $\mathcal{J}$, at every σ in $\mathcal{S}$,

$$u^i\left(z^i_\sigma\right) - u^i\left(x^i_\sigma\right) \geq \partial u^i\left(x^i_\sigma\right)\left(z^i_\sigma - x^i_\sigma\right) - \left(\frac{1-\rho}{2}\right)\partial u^i\left(x^i_\sigma\right)\left|z^i_\sigma - x^i_\sigma\right|.$$

This, because of condition (**), shows weak Pareto dominance. By strict convexity of preferences, this suffices to prove the claim. □

Proof of proposition 5. As allocation z lies in $X^{\text{PC}}(x)$, for every individual i in $\mathcal{J}$, at every σ in $\mathcal{S}$,

$$v^i_\sigma = \frac{1}{p^i_\sigma}\sum_{\tau\in\mathcal{S}(\sigma)} p^i_\tau\left(z^i_\tau - x^i_\tau\right) \geq 0.$$

In addition, $v = \sum_{i\in\mathcal{J}} v^i$ is a non-null positive element of $\mathbb{R}^{\mathcal{S}}$, as welfare is higher for at least one individual at some date-event. By feasibility and the bound on subjective prices (UI), as a matter of fact, for every individual i in $\mathcal{J}$, v^i is a positive element of L and, across individuals, v is a non-null positive element of L.

Observe that, by construction, for every individual i in $\mathcal{J}$, at every σ in $\mathcal{S}$,

$$\frac{1}{p^i_\sigma}\sum_{\tau\in\sigma_+} p^i_\tau v^i_\tau + \left(z^i_\sigma - x^i_\sigma\right) = v^i_\sigma.$$

By first-order conditions (FOC-1)-(FOC-2) and the positivity of v^i in L,

$$\frac{1}{p_\sigma}\sum_{\tau\in\sigma_+} p_\tau v^i_\tau + \left(z^i_\sigma - x^i_\sigma\right) \geq v^i_\sigma.$$

Summing among individuals,

$$\text{(*)}\qquad \frac{1}{p_\sigma}\sum_{\tau\in\sigma_+} p_\tau v_\tau + \sum_{i\in\mathcal{J}}\left(z^i_\sigma - x^i_\sigma\right) \geq v_\sigma.$$

We here distinguish two cases.

Assuming constrained inefficiency, condition (*) delivers, at every date-event σ in $\mathcal{S}$,

$$\frac{1}{p_\sigma}\sum_{\tau\in\sigma_+} p_\tau v_\tau \geq v_\sigma.$$

This proves the claim.

Assuming strongly constrained inefficiency, observe that, at every σ in $\mathcal{S}$, $v_\sigma > 0$ only if σ belongs to $\mathcal{F}$. (Indeed, if σ is not in $\mathcal{F}$, then, for every individual i in $\mathcal{J}$,

$$\left(z^i_\tau\right)_{\tau\in\mathcal{S}(\sigma)} = \left(x^i_\tau\right)_{\tau\in\mathcal{S}(\sigma)}$$

and, hence, $v^i_\sigma = 0$.) Therefore, as v is a bounded element in L, for some sufficiently large $1 > \rho > 0$, at every σ in $\mathcal{F}$,

$$\epsilon \geq \left(\frac{1-\rho}{\rho}\right) v_\sigma.$$

Hence, condition (*) delivers, at every date-event σ in $\mathcal{F}$,

$$\frac{1}{p_\sigma}\sum_{\tau\in\sigma_+} p_\tau v_\tau \geq v_\sigma + \epsilon \geq \left(\frac{1}{\rho}\right) v_\sigma.$$

Finally, at every date-event σ in $(\mathcal{S}/\mathcal{F})$, condition (*) delivers

$$\frac{1}{p_\sigma}\sum_{\tau\in\sigma_+} p_\tau v_\tau \geq 0 \geq \left(\frac{1}{\rho}\right) v_\sigma.$$

This proves the claim. □

Proof of proposition 6. By proposition 4, allocation x in X is constrained inefficient at initial allocation x in X. As allocation x lies in $X^{\text{PC}}(e)$, this simple observation suffices to prove the claim. □

Proof of proposition 7. For every individual i in $\mathcal{J}$, define, at every date-event σ in $\mathcal{S}$,

$$v^i_\sigma = \frac{1}{p^i_\sigma}\sum_{\tau\in\mathcal{S}(\sigma)} p^i_\tau\left(z^i_\tau - x^i_\tau\right).$$

By Pareto dominance, at the initial date-event ϕ in $\mathcal{S}$, $\sum_{i\in\mathcal{J}} v^i_\phi > 0$. In addition, as allocation x in X is interior, by uniform impatience (UI), for every individual i in $\mathcal{J}$, v^i is an element of L. In addition, for every individual i in $\mathcal{J}$, at every σ in $\mathcal{S}$,

$$(*)\qquad \frac{1}{p^i_\sigma}\sum_{\tau\in\sigma_+} p^i_\tau v^i_\tau + \left(z^i_\sigma - x^i_\sigma\right) \geq v^i_\sigma.$$

For every individual i in $\mathcal{J}$, at every σ in $\mathcal{S}$,

$$v^i_\sigma < 0 \text{ implies } U^i_\sigma\left(z^i\right) - U^i_\sigma\left(x^i\right) < 0.$$

Therefore, as allocation z lies in $X^{\text{PC}}(\nu)$,

$$v^i_\sigma < 0 \text{ implies } U^i_\sigma\left(x^i\right) - \nu^i_\sigma > 0.$$

Using the consistency requirement (DC), this yields

$$v^i_\sigma < 0 \text{ implies } f^i_\sigma > 0.$$

Hence, by first-order conditions (FOC-1)-(FOC-2), inequality (*) delivers, for every individual i in $\mathcal{J}$, at every σ in $\mathcal{S}$,

$$\frac{1}{p_\sigma}\sum_{\tau\in\sigma_+} p_\tau v^i_\tau + \left(z^i_\sigma - x^i_\sigma\right) \geq v^i_\sigma.$$

Summing up across individuals,

$$\frac{1}{p_\sigma}\sum_{\tau\in\sigma_+} p_\tau v_\tau + \sum_{i\in\mathcal{J}}\left(z^i_\sigma - x^i_\sigma\right) \geq v_\sigma.$$

We only consider the case of strongly constrained inefficiency, as the proof for simply constrained inefficiency is analogous.

The positive part v^+ of v in L is a non-null positive element of L. At a date-event σ in $\mathcal{S}$, if $v^+_\sigma > 0$, then σ in an element of $\mathcal{F}$. Hence, for some sufficiently large $1 > \rho > 0$, at every σ in $\mathcal{S}$,

$$\epsilon \geq \left(\frac{1-\rho}{\rho}\right) v^+_\sigma.$$

Therefore, at a date-event σ in $\mathcal{S}$, if $v^+_\sigma > 0$, then

$$\frac{1}{p_\sigma}\sum_{\tau\in\sigma_+} p_\tau v^+_\tau \geq v^+_\sigma + \epsilon \geq \left(\frac{1}{\rho}\right) v^+_\sigma.$$

In addition, at a date-event σ in $\mathcal{S}$, if $v_\sigma^+ = 0$, then

$$\frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau v_\tau^+ \geq 0 \geq \left(\frac{1}{\rho} \right) v_\sigma^+.$$

This suffices to prove the claim. □

Gaetano Bloise, Department of Economics, University of Rome III, Italy
E-mail address: gaetano.bloise@uniroma3.it

Pietro Reichlin, Department of Economics, LUISS 'Guido Carli', Rome, Italy; CEPR; EIEF
E-mail address: preichlin@luiss.it