INDETERMINACY OF COMPETITIVE EQUILIBRIUM WITH RISK OF DEFAULT

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Abstract. We prove indeterminacy of competitive equilibrium in sequential economies, where limited commitment requires the endogenous determination of solvency constraints preventing debt repudiation (Alvarez and Jermann [3]). In particular, we show that, for any arbitrary value of social welfare in between autarchy and (constrained) optimality, there exists an equilibrium attaining that value. Our method consists in restoring Welfare Theorems for a weak notion of (constrained) optimality. The latter, inspired by Malinvaud [15], corresponds to the absence of Pareto improving feasible redistributions over finite (though indefinite) horizons.

Keywords. Limited commitment; solvency constraints; Malinvaud efficiency; Welfare Theorems; indeterminacy; financial fragility; market collapse.

JEL Classification Numbers. D50, D52, D61, E44, G13.

1. Introduction

In this paper, we consider a large class of sequential economies with limited commitment over an infinite horizon under uncertainty. Asset markets are sequentially complete and endogenous solvency constraints prevent debt repudiation at equilibrium. In particular, as in Kehoe and Levine [10, 11], Kocherlakota [12] and Alvarez and Jermann [3, 4], traders might only borrow up to the point at which they are indifferent between honoring their debt obligations and reverting to permanent autarchy. The notion of competitive equilibrium is inherited from Alvarez and Jermann [3]. Accordingly, debt limits are taken as given by individuals and they are the largest values such that repayment is always individually rational (*i.e.*, they are not-too-tight).

A relevant feature of equilibria with not-too-tight debt constraints is that they may be (constrained) inefficient. This happens when the equilibrium price sequence involves low enough interest rates. In particular, the autarchic allocation can always be decentralized as an equilibrium and it is (constrained) inefficient when the marginal rate of substitution between present and future consumption of unconstrained individuals is sufficiently low.

Another (perhaps less known) feature of equilibrium with not-too-tight debt constraints is that it might be indeterminate. A classic example is the stationary symmetric two-agent economy with cyclic individual endowments (similar to an

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example provided by Bewley [7], recently elaborated in the limited commitment framework by Azariadis [5]). In this economy, autarchy is (constrained) inefficient when the high-endowment is relatively large with respect to the low-endowment, that is, when agents are very far from consumption smoothing in the absence of financial markets. In this case, there exists a unique (constrained) efficient equilibrium different from autarchy, allowing agents to get as close as possible to consumption smoothing, and a continuum of (constrained) inefficient non-stationary equilibrium allocations converging to autarchy. This type of examples suggests that there is a tight relation between inefficiency and indeterminacy.

We show that this conjecture can be made a precise statement (and can formally be proved) in the following sense. Given any arbitrary value of social welfare in between autarchy and (constrained) optimality, there exists an equilibrium with not-too-tight debt constraints attaining that value. In other terms, there is a continuum of equilibria with welfare declining from (constrained) efficiency to autarchy.

We adapt the canonical method based on Welfare Theorems to characterize the set of competitive equilibria. In particular, we introduce a weak form of (constrained) optimality: Malinvaud, or short-run, optimality corresponds to the absence of a feasible welfare-improving reallocation restricted to a finite number of periods (as in Malinvaud [15, 16], Balasko and Shell [6] and Aliprantis, Brown and Burkinshaw [2]). More intuitively, it is achieved when allocations satisfy the canonical first-order conditions, or Euler equations, for a social planner problem, though a social condition of transversality might fail. This criterion of optimality is weak in the sense that, whereas an efficient allocation is always Malinvaud optimal, inefficient allocations (autarchy included) may be Malinvaud optimal. We then show that any equilibrium is a Malinvaud optimum (First Welfare Theorem) and, conversely, any Malinvaud optimum can be sustained as an equilibrium for some balanced distribution of initial assets (Second Welfare Theorem). Malinvaud optima, in turn, can be generated as limits of solutions to artificial social planner problems under the restrictions imposed by the notion of short-run optimality. In fact, an approximation method allows us to prove that there exists a large set of Malinvaud optimal allocations with social welfare declining from (constrained) optimality to autarchy. By Welfare Theorems, this structure is inherited by the set of equilibrium allocations under limited commitment. Hence, equilibria are globally indeterminate. Limited commitment produces an unavoidable fragility of financial markets, leading to a complete collapse (autarchy).

Indeterminacy of equilibria might be understood as the consequence of a dynamic complementarity between current and expected future credit constraints. When individuals expect a fall in future debt limits (*i.e.*, they believe to be less likely to smooth out consumption through asset markets), the current value of participation goes down and incentives to default increase. Since current debt limits adjust endogenously to market conditions, they fall immediately as a response to lower participation incentives (or a loss of reputation). Indeterminacy is produced by a failure of social transversality (*i.e.*, low implied interest rates), as in overlapping generations economies, where a change in expectations might lead to a contraction of trades (across generations) and a convergence to autarchy. The relation between indeterminacy and inefficiency is there controversial (see, for instance, Kehoe and Levine [9]).

The paper is organized as follows. In sections 2 and 3, we lay out the fundamentals of a general multi-agent economy with uncertainty and we define a notion of competitive equilibrium with sequential trades and not-too-tight debt constraints. In section 4, we present our Indeterminacy Theorem. In section 5, we introduce Malinvaud efficiency and provide a partial characterization of Malinvaud optima. In section 6, we establish Welfare Theorems relatively to this weak form of efficiency. All proofs are collected in the appendix.

2. Fundamentals

2.1. **Time and uncertainty.** Time and uncertainty are represented by an event-tree $\mathcal{S}$, a countably infinite set, endowed with ordering $\succeq$. For a date-event σ in $\mathcal{S}$, $t(\sigma)$ in $\mathcal{T} = \{0, 1, 2, \ldots, t, \ldots\}$ denotes its date and

$$\sigma_+ = \{\tau \in \mathcal{S}(\sigma) : t(\tau) = t(\sigma) + 1\}$$

is the non-empty finite set of all immediate direct successors, where

$$\mathcal{S}(\sigma) = \{\tau \in \mathcal{S} : \tau \succeq \sigma\}$$

is the set of all date-events τ in $\mathcal{S}$ (weakly) following date-event σ in $\mathcal{S}$. The initial date-event is ϕ in $\mathcal{S}$, with $t(\phi) = 0$, that is, $\sigma \succeq \phi$ for every σ in $\mathcal{S}$; the initial date-event in $\mathcal{S}(\sigma)$ is σ in $\mathcal{S}$. This construction is canonical (Debreu [8, Chapter 7]).

2.2. **Vector spaces.** We essentially adhere to Aliprantis and Border [1, Chapters 5-8] for terminology and notation. The reference vector space is $L = \mathbb{R}^{\mathcal{S}}$, the space of all real-valued maps on $\mathcal{S}$, with typical element

$$v = (v_\sigma)_{\sigma \in \mathcal{S}}.$$

The vector space L is endowed with the canonical order: an element v of L is *positive* if $v_\sigma \geq 0$ for every σ in $\mathcal{S}$; it is *strictly positive* if $v_\sigma > 0$ for every σ in $\mathcal{S}$; finally, it is *uniformly strictly positive* if, for some $\epsilon > 0$, $v_\sigma \geq \epsilon$ for every σ in $\mathcal{S}$. For a positive element v of L, we simply write $v \geq 0$ and, when v in L is also non-null, $v > 0$. Finally, the positive cone of any (Riesz) vector subspace F of L is $\{v \in F : v \geq 0\}$.

For an element v of L, v^+ in L and v^- in L are, respectively, its positive part and its negative part, so that $v = v^+ - v^-$ in L and $|v| = v^+ + v^-$ in L. Also, for an arbitrary collection $\{v^j\}_{j \in \mathcal{J}}$ of elements of L, its supremum and its infimum in L, if they exist, are denoted, respectively, by

$$\bigvee_{j \in \mathcal{J}} v^j \text{ and } \bigwedge_{j \in \mathcal{J}} v^j.$$

An element v of L is *bounded* if, for some $\epsilon > 0$, $|v_\sigma| \leq \epsilon$ for every σ in $\mathcal{S}$; it is *summable* if

$$\sum_{\sigma \in \mathcal{S}} |x_\sigma| \text{ is finite;}$$

it is *eventually vanishing* if $\{\sigma \in \mathcal{S} : |v_\sigma| > 0\}$ is a finite subset of $\mathcal{S}$. The (Riesz) vector subspace of L, consisting of all eventually vanishing elements v of L, is denoted by C. Finally, unless otherwise explicitly stated, the vector space L is endowed with the product topology.

2.3. **Individuals.** There is a finite set $\mathcal{J}$ of individuals. For every individual i in $\mathcal{J}$, the *consumption space* X^i is the positive cone of the *commodity space* L. A consumption plan x^i in X^i is *interior* (respectively, *bounded*) if it is uniformly strictly positive (respectively, bounded). An *allocation* is a distribution of consumption plans across individuals. The space of allocations is

$$X = \left\{ x \in L^{\mathcal{J}} : x^i \in X^i \text{ for every } i \in \mathcal{J} \right\}.$$

An allocation x in X is *interior* (respectively, *bounded*) if every consumption plan x^i in X^i is interior (respectively, bounded).

2.4. **Endowments.** For every individual i in $\mathcal{J}$, the *endowment* e^i in X^i is interior and bounded. In particular, there exists a sufficiently small $1 > \epsilon > 0$ satisfying, at every date-event σ in $\mathcal{S}$,

$$\epsilon \leq \bigwedge_{i \in \mathcal{J}} e^i_\sigma \leq (\#\mathcal{J}) \bigvee_{i \in \mathcal{J}} e^i_\sigma \leq \frac{1}{\epsilon}.$$

This hypothesis imposes a uniform lower bound on the endowment of individuals and, across individuals, an upper bound on the aggregate endowment.

2.5. **Preferences.** For every individual i in $\mathcal{J}$, the per-period utility function $u^i : \mathbb{R}_+ \to \mathbb{R}$ is bounded, continuous, continuously differentiable, strictly increasing and strictly concave. (As far as smoothness is concerned, more precisely, the per-period utility function is continuously differentiable on $\mathbb{R}_{++}$.) For every individual i in $\mathcal{J}$, the *utility function* $U^i : X^i \to \mathbb{R}$ is given by

$$U^i \left(x^i \right) = \sum_{\sigma \in \mathcal{S}} \pi^i_\sigma u^i \left(x^i_\sigma \right),$$

where π^i is a strictly positive summable element of L. Also, for any date-event σ in $\mathcal{S}$, at any consumption plan x^i in X^i,

$$U^i_\sigma \left(x^i \right) = \frac{1}{\pi^i_\sigma} \sum_{\tau \in \mathcal{S}(\sigma)} \pi^i_\tau u^i \left(x^i_\tau \right).$$

This is the continuation utility beginning from date-event σ in $\mathcal{S}$.

2.6. **Uniform impatience.** We impose a uniform bound on the marginal rate of substitution of perpetual future consumption for current consumption. This hypothesis implies a uniform form of impatience across individuals and date-events. Basically, there exists a sufficiently small $1 > \eta > 0$ satisfying, for every individual i in $\mathcal{J}$, at every date-event σ in $\mathcal{S}$,

$$\pi^i_\sigma \geq \eta \sum_{\tau \in \mathcal{S}(\sigma)} \pi^i_\tau.$$

2.7. **(Weak) Inada conditions.** This additional hypothesis serves to ensure interiority. For every individual i in $\mathcal{J}$, at every date-event σ in $\mathcal{S}$,

$$\eta u^i (0) + (1 - \eta) u^i \left(\frac{1}{\epsilon} \right) < u^i (\epsilon),$$

where $1 > \epsilon > 0$ is given by the bounds on endowments and $1 > \eta > 0$ by the hypothesis of uniform impatience.

2.8. **Subjective prices.** At an interior consumption plan x^i in X^i, the subjective price p^i in P^i is defined by

$$\left(p^i_\sigma\right)_{\sigma \in \mathcal{S}} = \left(\pi^i_\sigma \partial u^i \left(x^i_\sigma\right)\right)_{\sigma \in \mathcal{S}}.$$

The subjective price p^i in P^i is a strictly positive summable element of L.

2.9. **Feasible allocations.** An allocation x in X is *feasible* if it exhausts aggregate resources and satisfies participation constraints, that is,

$$\sum_{i \in \mathcal{J}} x^i = \sum_{i \in \mathcal{J}} e^i$$

and, for every individual i in $\mathcal{J}$, at every date-event σ in $\mathcal{S}$,

$$U^i_\sigma \left(x^i\right) \geq U^i_\sigma \left(e^i\right).$$

The space of all feasible allocations is denoted by $X(e)$. Notice that feasibility reflects both material constraints and participation constraints.

Under the maintained assumptions on preferences and endowments, every feasible allocation is, as a matter of fact, an interior allocation. The particular form of (weak) Inada conditions, which is a joint restriction on preferences and endowments, guarantees interiority of consumptions, subject to participation constraints, avoiding unbounded per-period utilities and, hence, simplifying the presentation.

Lemma 1 (Interiority)**.** *Every feasible allocation is interior.*

3. Equilibrium

Trade occurs sequentially. In every period of trade, given revealed uncertainty, a full spectrum of elementary Arrow securities is available, yielding unitary payoffs in the following period of trade, contingent on the occurrence of events. The asset market is, thus, sequentially complete. It simplifies to represent implicit prices of contingent commodities in terms of present values. They are denoted by p in P, the space of all strictly positive elements of L. At every date-event σ in $\mathcal{S}$, a portfolio, with deliveries v in L at the following date-events, has a market value, in terms of current consumption, given by

$$\frac{1}{p_\sigma} \sum_{\tau \in \sigma_+} p_\tau v_\tau.$$

An individual i in $\mathcal{J}$ participates into financial markets. The holding of securities is represented by a *financial plan* v^i in V^i, the space of all unrestricted elements of L. Positive values correspond to claims, whereas negative values are liabilities. This participation occurs subject to sequential budget constraints, imposing, at every date-event σ in $\mathcal{S}$,

$$\sum_{\tau \in \sigma_+} p_\tau v^i_\tau + p_\sigma \left(x^i_\sigma - e^i_\sigma\right) \leq p_\sigma v^i_\sigma.$$

Accumulated wealth serves to finance current consumption, in excess to current endowment, and current net asset positions (claims or liabilities). Participation into financial markets is further restricted by quantitative limits to private liabilities. These *debt limits* are given by f^i in F^i, the set of all positive and bounded elements of L. Individual i in $\mathcal{J}$ can issue debt obligations subject to debt constraints given, at every date-event σ in $\mathcal{S}$, by

$$-f_\sigma^i \leq v_\sigma^i.$$

From the perspective of the individual, these debt limits are given exogenously.

As in Kehoe and Levine [10], Kocherlakota [12] and Alvarez and Jermann [3], commitment is limited. Individuals might not honor their debt obligations, even though the material availability of future endowments would suffice for a complete repayment. When debt is repudiated, assets are seized and the individual is excluded from future participation into financial markets, though maintaining claims into future uncertain endowment. Thus, unhonored debt induces a permanent reverse to autarchy. At equilibrium, debt limits serve to guarantee that, on the one side, debt repudiation is not profitable for individuals and, on the other side, the maximum sustainable development of financial markets is enforced. This is the notion of equilibrium with *not-too-tight* debt constraints provided by Alvarez and Jermann [3].

Formally, an allocation x in X is an *equilibrium allocation* if there exist a price p in P, debt limits f in F and financial plans v in V satisfying the following properties:

(a) For every individual i in $\mathcal{J}$, the plan $\left(x^i, v^i\right)$ in $X^i \times V^i$ is optimal subject to budget and debt constraints, given initial claims, that is, it maximizes intertemporal utility subject, at every date-event σ in $\mathcal{S}$, to budget constraint,

$$\sum_{\tau \in \sigma_+} p_\tau \bar{v}_\tau^i + p_\sigma \left(\bar{x}_\sigma^i - e_\sigma^i\right) \leq p_\sigma \bar{v}_\sigma^i,$$

and to debt constraints,

$$-\left(\bar{v}_\tau^i + f_\tau^i\right)_{\tau \in \sigma_+} \leq 0,$$

given initial wealth v_ϕ^i in $\mathbb{R}$.

(b) Commodity and financial markets clear, that is,

$$\sum_{i \in \mathcal{J}} x^i = \sum_{i \in \mathcal{J}} e^i \text{ and } \sum_{i \in \mathcal{J}} v^i = 0.$$

(c) For every individual i in $\mathcal{J}$, debt limits are not-too-tight, that is, at every date-event $\bar{\sigma}$ in $\mathcal{S}$,

$$J_{\bar{\sigma}}^i \left(-f_{\bar{\sigma}}^i; f^i\right) = U_{\bar{\sigma}}^i \left(e^i\right),$$

where

$$J_{\bar{\sigma}}^i \left(w_{\bar{\sigma}}^i; f^i\right) = \sup U_{\bar{\sigma}}^i \left(\bar{x}^i\right)$$

subject, at every date-event σ in $\mathcal{S}(\bar{\sigma})$, to budget constraint,

$$\sum_{\tau \in \sigma_+} p_\tau \bar{v}_\tau^i + p_\sigma \left(\bar{x}_\sigma^i - e_\sigma^i\right) \leq p_\sigma \bar{v}_\sigma^i,$$

and to debt constraints,

$$-\left(\bar{v}_\tau^i + f_\tau^i\right)_{\tau \in \sigma_+} \leq 0,$$

given initial wealth $w_{\bar{\sigma}}^i$ in $\mathbb{R}$. (By convention, the supremum over an empty set is negative infinity.)

Notice that, at equilibrium, for every individual i in $\mathcal{J}$, at every date-event σ in $\mathcal{S}$,

$$U_\sigma^i\left(x^i\right) = J_\sigma^i\left(v_\sigma^i; f^i\right) \geq J_\sigma^i\left(-f_\sigma^i; f^i\right) = U_\sigma^i\left(e^i\right).$$

Hence, an equilibrium allocation x in X is, as a matter of fact, an element of $X(e)$, the space of feasible allocations.

We adopt a restrictive notion of equilibrium: first, we require debt limits to be positive and bounded; second, we exclude speculative bubbles. Negative debt limits, that are allowed by Alvarez and Jermann [3], would impose to individuals the holding of positive wealth along some contingencies, an unnatural requirement in our view. Kocherlakota [13] shows some properties of homogeneity of the budget set. Negative and unbounded debt limits would sustain speculative bubbles at equilibrium. Also, notice that, at equilibrium, for every individual i in $\mathcal{J}$, debt limits f^i in F^i need be *consistent* (according to the terminology borne out by Levine and Zame [14]), that is, at every date-event σ in $\mathcal{S}$,

$$p_\sigma f_\sigma^i \leq p_\sigma e_\sigma^i + \sum_{\tau \in \sigma_+} p_\tau f_\tau^i.$$

Hence, the maximum amount of debt can be honored by means of current endowment and by issuing future debt up the maximum amount.

4. Indeterminacy

Debt contracts are enforced by the threat of exclusion from financial markets and might sustain (limited) risk-sharing at equilibrium. However, the underlying mechanism is merely reputational and, in a sense, fragile. Competitive equilibrium is indeterminate.

We relate multiplicity of equilibria to social welfare. Given welfare weights θ in Θ, social welfare, at allocation x in X, is measured by the weighted sum of utilities,

$$W_\theta(x) = \sum_{i \in \mathcal{J}} \theta^i U^i\left(x^i\right),$$

where

$$\Theta = \left\{\theta \in \mathbb{R}_+^{\mathcal{J}} : \sum_{i \in \mathcal{J}} \theta^i = 1\right\}.$$

Efficient values obtain when the planner maximizes social welfare subject to feasibility, encompassing material and participation constraints, that is,

$$W_\theta^* = \max_{z \in X(e)} W_\theta(z).$$

Clearly, when autarchy is inefficient, $W_\theta^* > W_\theta(e)$ for all welfare weights θ in Θ.

Indeterminacy Theorem. *Given welfare weights θ in Θ, for any arbitrary value ξ in $\Xi = [0,1]$, there exists an equilibrium allocation x in $X(e)$ with social welfare satisfying*

$$W_\theta(x) = \xi W_\theta^* + (1-\xi) W_\theta(e).$$

Welfare weights account for a merely distributive multiplicity, typically reflecting the allocation of initial claims inherited from the unrepresented past. The index ξ in Ξ, instead, measures the degree of market confidence, or of market soundness, or of credit expansion, decreasing from the maximum sustainable development of financial markets (efficiency) to the complete collapse of financial markets (autarchy).

Equilibrium exhibits a global form of indeterminacy. Though debt limits are generated by fundamentals by means of participation constraints, financial fragility is an intrinsically unavoidable phenomenon.

To prove the Indeterminacy Theorem, we amend the classical method of analysis that exploits Welfare Theorems. In particular, we introduce a weak form of efficiency, referred to as Malinvaud efficiency. This requires the absence of welfare-improving feasible redistributions over finite horizons only. We then show that Malinvaud efficient allocations form a large set, with social welfare decreasing from efficiency to autarchy. We finally prove that any equilibrium allocation is Malinvaud efficient (First Welfare Theorem) and that any Malinvaud efficient allocation emerges as an equilibrium allocation for some distribution of initial claims across individuals (Second Welfare Theorem). As a matter of fact, the multiplicity of Malinvaud optima reflects upon competitive equilibrium.

5. Malinvaud Optima

5.1. **Malinvaud efficiency.** Malinvaud efficiency is inherited from studies on capital theory (*e.g.*, Malinvaud [15, 16]) and overlapping generations economies (*e.g.*, Balasko and Shell [6]). The canonical notion of Pareto efficiency requires the absence of a welfare improvement, subject to material and participation constraints. Thus, an allocation x in $X(e)$ is *Pareto efficient* if it is not Pareto dominated by an alternative allocation z in $X(e)$. The notion of Malinvaud efficiency, instead, imposes weaker restrictions, as it simply requires the absence of a welfare improvement, subject to material and participation constraints, only over any arbitrary finite horizon. Consistently, an allocation x in $X(e)$ is *Malinvaud efficient* if it is not Pareto dominated by an alternative allocation z in $X(e) \cap C(x)$, where

$$C(x) = \left\{ z \in X : \sum_{i \in \mathcal{J}} \left| z^i - x^i \right| \in C \right\}$$

is the set of all allocations z in X that modify allocation x in X only over a finite horizon. (Remember that C is the set of all eventually vanishing elements of L.) Clearly, any Pareto optimum is a Malinvaud optimum. However, Malinvaud optimality is a largely weaker requirement: for instance, any autarchic allocation is a Malinvaud optimum.

Malinvaud efficiency admits a characterization in terms of supporting price. This is an elaboration on the common duality argument, developed in the literature on capital theory and, more recently, for economies of overlapping generations by Aliprantis, Brown and Burkinshaw [2]. The (algebraic) dual of the vector subspace C of L can be identified with L itself, under the duality operation given, for every (v, f) in $C \times L$, by

$$f(v) = f \cdot v = \sum_{\sigma \in \mathcal{S}} f_\sigma v_\sigma .$$

Lemma 2 (First-order conditions)**.** *An allocation x in $X(e)$ is Malinvaud efficient if and only if there exists a price p in P satisfying, at every allocation z in $X^*(e) \cap C(x)$, for every individual i in $\mathcal{J}$,*

$$U^i\left(z^i\right) > U^i\left(x^i\right) \text{ only if } p \cdot \left(z^i - x^i\right) > 0, \tag{s}$$

where $X^*(e)$ is the set of all allocations z in X such that, for every individual i in $\mathcal{J}$, at every date-event σ in $\mathcal{S}$,

$$U_\sigma^i\left(z^i\right) \geq U_\sigma^i\left(e^i\right).$$

Equivalently, an allocation x in $X(e)$ is Malinvaud efficient if and only if there exists a price p in P satisfying, for every individual i in $\mathcal{J}$, at every date-event σ in $\mathcal{S}$,

$$\text{(C-1)} \qquad \left(\frac{p_\tau}{p_\sigma}\right)_{\tau\in\sigma_+} \geq \left(\frac{p_\tau^i}{p_\sigma^i}\right)_{\tau\in\sigma_+}$$

and

$$\text{(C-2)} \qquad \sum_{\tau\in\sigma_+}\left(\frac{p_\tau}{p_\sigma}\right)\left(U_\tau^i\left(x^i\right)-U_\tau^i\left(e^i\right)\right) = \sum_{\tau\in\sigma_+}\left(\frac{p_\tau^i}{p_\sigma^i}\right)\left(U_\tau^i\left(x^i\right)-U_\tau^i\left(e^i\right)\right),$$

where p^i in P^i is the subjective price at interior consumption plan x^i in X^i.

Restriction (S) coincides with an admittedly abstract characterization of Malinvaud optima in terms of supporting positive linear functionals, whereas conditions (C-1)-(C-2) uncover an equivalent formulation in terms of more treatable first-order conditions. For the sake of simplicity, the above characterization might be interpreted as referring to a canonical social planner problem. Restrictions (C-1)-(C-2) correspond, in this analogy, to the Euler equations induced by the maximization of (weighted) social welfare subject to material constraints and to participation constraints. They basically rule out the circumstance of a constrained individual exhibiting a marginal rate of substitution strictly above the marginal rate of substitution of an unconstrained individual. This, indeed, would expose to an arbitrage opportunity, as a substitution of future consumption for current consumption of the unconstrained individual, balanced by the opposite substitution for the constrained individual, would not violate participation constraint, as utility of the unconstrained individual is strictly above the autarchic utility, and would produce a welfare improvement. The remarkable implication of this full characterization is that a Malinvaud optimum does not impose any restriction in terms of social transversality or, alternatively, does not rule out any arbitrage opportunity *at infinitum.* A substitution of current consumption for perpetual future consumption might still generate a welfare improvement, subject to feasibility.

5.2. **Multiplicity.** We here provide a partial characterization of Malinvaud optima. In particular, we prove that there exists a continuum of such optima with social welfare decreasing from Pareto efficiency to autarchy. (Obviously, when the autarchy is Pareto efficient, this multiplicity disappears.) Malinvaud optima are parameterized by welfare weights θ in Θ and an index ξ in $\Xi = [0,1]$ measuring the failure of Pareto optimality. Hence, the set of Malinvaud optima contains a set that is isomorphic to $\Theta \times \Xi$.

Proposition 1 (Multiplicity)**.** *Given welfare weights θ in Θ, for any arbitrary value ξ in $\Xi = [0,1]$, there exists a Malinvaud efficient allocation x in $X(e)$ with social welfare satisfying*

$$(*) \qquad W_\theta(x) = \xi W_\theta^* + (1-\xi)W_\theta(e).$$

The difficulty for the understanding of the structure of Malinvaud optima stems from the fact that they cannot be directly obtained as solutions of a well-defined social planning programme. This notwithstanding, a very simple characterization emerges by means of artificial truncated planner problems, along with a limit argument. These truncations obtain by imposing additional restrictions on the amount of redistributed resources that can be implemented out of some finite horizon. For a given truncation, the severity of these additional restrictions determines the value of the social planner problem: under the most severe restrictions, the redistribution vanishes out of a finite horizon and, hence, the autarchy is the only feasible allocation (indeed, a decrease of consumption in the last period of the truncation cannot be compensated by an increase of consumption in the following periods and, hence, by induction, no redistribution is the only feasible policy); under the least severe restrictions, any feasible allocation can be implemented and, hence, a Pareto optimum obtains. It follows that, for any given truncation, some properly chosen degree of severity of additional constraints would yield a given social welfare in between autarchy and Pareto efficiency. Taking the limit over finite horizons, a limit allocation emerges with a given social welfare value (as this can be assumed to be constant along the sequence). This limit allocation is Malinvaud efficient because, as the finite horizon extends along the sequence of truncations, first-order conditions are satisfied along larger and larger horizons. We remark that other forms of truncations are practicable and would deliver analogous conclusions: for instance, adding restrictions only beginning from some contingency or distributing restrictions across contingencies. Moreover, we believe that an analogous method could prove it applicable in other economies exhibiting a failure of social transversality (for instance, for a global characterization of competitive equilibria in economies of overlapping generations).

6. Welfare Theorems

We here show equivalence between equilibrium allocations and Malinvaud efficient allocations. Indeed, any equilibrium allocation is Malinvaud efficient (First Welfare Theorem) and any Malinvaud efficient allocation emerges as an equilibrium allocation for some balanced distribution of initial claims (Second Welfare Theorem). As a matter of fact, we prove that Malinvaud efficiency exhausts all restrictions on equilibrium prices and allocations.

Proposition 2 (First Welfare Theorem)**.** *Any equilibrium allocation is a Malinvaud efficient allocation.*

The First Welfare Theorem is almost immediate. Indeed, first-order conditions for a Malinvaud optimum coincides with those for an equilibrium under limited commitment (see Alvarez and Jermann [3]). At equilibrium, the marginal rate of substitution of an individual falls below the market rate of substitution only if this individual is constrained in issuing further debt obligations, for otherwise a budget-balanced (marginal) substitution of future consumption for current consumption would yield an increase in welfare.

Proposition 3 (Second Welfare Theorem)**.** *Any Malinvaud efficient allocation is an equilibrium allocation.*

The proof of the Second Welfare Theorem cannot rely on a traditional separation argument alone. Indeed, separation yields potential equilibrium prices fulfilling first-order conditions (lemma 2). Such prices, however, might not belong to the dual of the commodity space (restricted by the aggregate endowment) and, thus, might not deliver a well-defined intertemporal accounting. In order to provide their Second Welfare Theorem for Pareto efficient allocations, Alvarez and Jermann [3] assume that prices belong to the dual of the (restricted) commodity space (the hypothesis of high implied interest rates) and recover financial plans at equilibrium as the present value of future contingent net trades. We cannot count on this simple method and need an alternative proof. Furthermore, differently from Alvarez and Jermann [3], as well as from Kocherlakota [13], we impose positivity of debt limits (individuals cannot be restricted to hold positive amounts of wealth along the infinite horizon), which poses additional difficulties.

To recover financial plans, we move from a basic observation. We evaluate welfare gains, with respect to the autarchic utility, in terms of current consumption, by using marginal utilities. Participation guarantees that these welfare gains are positive across date-events. Also, they fulfill sequential budget constraints at subjective prices (marginal utilities). As market rates of substitution differ from individual marginal rates of substitution only when welfare gains vanish, the process of welfare gains also satisfies sequential budget constraints at market prices. This yields an upper bound on the amount of wealth held at equilibrium, as welfare gains are positive (hence, fulfil debt limits) and sustains the given consumption plan subject to sequential budget constraints. As financial plans need be balanced at equilibrium across individuals, the negative of the sum of welfare gains poses a lower bound to financial plans. Therefore, having identified a suitable interval, we construct an adjustment process that increases debt, when more debt is budget-feasible, and decreases debt, when outstanding debt is budget-unfeasible. This process admits a fixed point and, at the fixed point, sequential budget constraints are balanced and financial markets clear.

Optimality of consumption plans, subject to budget constraints and debt constraints, is ensured by first-order conditions at a Malinvaud optimum. Hence, it only remains to reconstruct suitable debt limits. Here, we follow Alvarez and Jermann [3]. When an individual is at the autarchic utility, outstanding debt coincides with the maximum amount of debt. When an individual is not at the autarchic utility, we compute the maximum amount of sustainable debt, which depends on the future contingent plan for debt limits. Beginning with sufficiently large debt limits, this process of adjustment generates a decreasing sequence of debt limits and, in the limit, we obtain not-too-tight debt constraints. The identification of suitable upper bounds requires some elaboration.

7. Conclusion

We have shown that equilibria of economies with limited enforcement and not-too-tight debt limits are indeterminate. In particular, we have developed a method that exploits Welfare Theorems for deriving a full characterization of equilibria. These theorems are established for a weak form of optimality, corresponding to the absence of a feasible Pareto improving redistribution over a finite number of time periods. These weak optima, in turn, are characterized by means of sequences of planning objectives with limited amounts of redistributions in the long-run. The method shows that, at equilibrium, social welfare ranges from two extreme outcomes: constrained Pareto optimality and autarchy.

This paper bears very important consequences on the understanding of the type of equilibria that may emerge in economies where contract enforcement is limited and the no default option is implemented by imposing individual specific debt constraints. In particular, these equilibria suffer from a severe form of financial fragility: a change in expectations at any given equilibrium, where asset trades guarantee an optimal amount of consumption smoothing across states and time periods, might generate a contraction of net trades, in some cases leading to financial collapse.

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Proofs

Proof of lemma 1. At a feasible allocation, for every individual i in $\mathcal{J}$, participation constraints impose, at every date-event σ in $\mathcal{S}$,

$$\begin{aligned} u^i\left(x^i_\sigma\right) - u^i\left(\frac{1}{\epsilon}\right) + \frac{1}{\pi^i_\sigma}\sum_{\tau\in\mathcal{S}(\sigma)} \pi^i_\tau u^i\left(\frac{1}{\epsilon}\right) &\geq \\ U^i_\sigma\left(x^i\right) &\geq U^i_\sigma\left(e^i\right) \\ &\geq \frac{1}{\pi^i_\sigma}\sum_{\tau\in\mathcal{S}(\sigma)} \pi^i_\tau u^i\left(\epsilon\right). \end{aligned}$$

Therefore, exploiting uniform impatience and (weak) Inada conditions,

$$u^i\left(x^i_\sigma\right) \geq u^i\left(\frac{1}{\epsilon}\right) + \frac{1}{\eta}\left(u^i(\epsilon) - u^i\left(\frac{1}{\epsilon}\right)\right) > u^i(0),$$

which produces a uniformly strictly positive lower bound on consumptions. □

Proof of lemma 2. Sufficiency of a supporting price p in P (*i.e.*, condition (S)) for Malinvaud efficiency is obvious, as it is proved by the traditional argument for the canonical First Welfare Theorem. Therefore, we show that restrictions (C-1)-(C-2) imply condition (S). Consider any alternative allocation z in $X^*(e) \cap C(x)$ and suppose that, for some individual i in $\mathcal{J}$,

$$0 < U^i\left(z^i\right) - U^i\left(x^i\right) \leq \sum_{\sigma \in \mathcal{S}} p^i_\sigma \left(z^i_\sigma - x^i_\sigma\right).$$

Define, at every date-event σ in $\mathcal{S}$,

$$v^i_\sigma = \frac{1}{p^i_\sigma} \sum_{\tau \in \mathcal{S}(\sigma)} p^i_\tau \left(z^i_\tau - x^i_\tau\right).$$

Notice that v^i is an element of C. A simple decomposition yields, at every date-event σ in $\mathcal{S}$,

$$(*) \qquad \sum_{\tau \in \sigma_+} p^i_\tau v^i_\tau + p^i_\sigma \left(z^i_\sigma - x^i_\sigma\right) \geq p^i_\sigma v^i_\sigma.$$

Furthermore, notice that convexity of preferences and participation constraints imply that, at every date-event σ in $\mathcal{S}$,

$$v^i_\sigma < 0 \text{ only if } U^i_\sigma\left(e^i\right) \leq U^i_\sigma\left(z^i\right) < U^i_\sigma\left(x^i\right).$$

Therefore, restrictions (C-1)-(C-2), along with inequality (*), guarantee that, at every date-event σ in $\mathcal{S}$,

$$\sum_{\tau \in \sigma_+} p_\tau v^i_\tau + p_\sigma \left(z^i_\sigma - x^i_\sigma\right) \geq p_\sigma v^i_\sigma.$$

Consolidating across date-events, and noticing that v^i is an element of C, one obtains

$$\begin{aligned} p \cdot \left(z^i - x^i\right) &= \sum_{\sigma \in \mathcal{S}} p_\sigma \left(z^i_\sigma - x^i_\sigma\right) \\ &\geq p_\phi v^i_\phi \\ &= \left(\frac{p_\phi}{p^i_\phi}\right) \sum_{\sigma \in \mathcal{S}} p^i_\sigma \left(z^i_\sigma - x^i_\sigma\right) \\ &> 0, \end{aligned}$$

thus proving the claim.

Assume now that the allocation x in $X(e)$ is Malinvaud-efficient and define a price p in P by means, at every date-event σ in $\mathcal{S}$, of

$$\left(\frac{p_\tau}{p_\sigma}\right)_{\tau \in \sigma_+} = \bigvee_{i \in \mathcal{J}} \left(\frac{p^i_\tau}{p^i_\sigma}\right)_{\tau \in \sigma_+}.$$

This price p in P obviously satisfies condition (C-1). The necessity of condition (C-2) straightforwardly obtains by means of the argument in Alvarez and Jermann

[3, Proposition 3.1]. As conditions (C-1)-(C-2) imply restriction (S), this completes the proof. □

Proof of proposition 1. The proof is decomposed in several separate steps. First, we construct a sequence of truncated planner problems, by adding additional auxiliary constraints on the transfers across individuals; truncated optima exist and, at given welfare weights θ in Θ, social welfare might be measured by ξ in Ξ by controlling for the severity of additional constraints on transfers. Second, we generate a sequence of truncated optima, maintaining a constant value of social welfare, and we consider the limit allocation of these truncated planner problems. Third, we prove that the limit allocation is in fact a Malinvaud optimum.

Truncation. Given any t in $\mathcal{T}$, consider a collection of t-truncated planner problems:

$$\max_{x \in X(e)} \sum_{i \in \mathcal{J}} \theta^i \left(U^i \left(x^i \right) - U^i \left(e^i \right) \right)$$

subject to, at every date-event σ in $(\mathcal{S}/\mathcal{S}^t)$,

$$(\dagger) \qquad \sum_{i \in \mathcal{J}} \left| x^i_\sigma - e^i_\sigma \right| \leq \epsilon,$$

where, for every t in $\mathcal{T}$,

$$\mathcal{S}^t = \{\sigma \in \mathcal{S} : t(\sigma) \leq t\}.$$

Constraints are given as a continuous correspondence of ϵ in $\mathbb{R}_+$ with non-empty convex and compact values. (Indeed, notice that the map $x \mapsto |x|$ is convex. In addition, if allocation x in $X(e)$ satisfies constraints (†) at ϵ in $\mathbb{R}_{++}$, then allocation

$$x - \left(\frac{\epsilon - \epsilon^*}{\epsilon} \right)^+ (x - e) \in X(e)$$

satisfies constraints (†) at ϵ^* in $\mathbb{R}_+$.) Hence, by the Maximum Theorem, the maximum is achieved and the value function is continuous in ϵ in $\mathbb{R}_+$.

Observe that, when ϵ in $\mathbb{R}_+$ is sufficiently large, the truncated problem delivers a Pareto efficient allocation; when ϵ in $\mathbb{R}_+$ vanishes, the truncated problem delivers the autarchy, as this is the only feasible allocation x in $X(e)$ satisfying additional constraints (†). Hence, by the Intermediate Value Theorem, restriction (*) is satisfied by some value of ϵ in $\mathbb{R}_+$. Let x^t be an allocation in $X(e)$ that solves the t-truncated planner problem at the value of ϵ in $\mathbb{R}_+$ fulfilling restriction (*). □

Limit. The sequence of allocation $\{x^t\}_{t \in \mathcal{T}}$ in $X(e)$, at no loss of generality, converges to some allocation x in $X(e)$ in the product topology. Also, by continuity of preferences, restriction (*) is satisfied by the limit allocation x in $X(e)$. □

Malinvaud optimality in the limit. We show that the limit allocation x in X is Malinvaud efficient. To this purpose, suppose that it is Pareto dominated by an alternative allocation z in $X(e) \cap C(x)$. For every individual i in $\mathcal{J}$, let $\mathcal{F}^i$ be the finite subset of all date-events in $\mathcal{S}$ at which the reallocation is not terminated, that is,

$$\sigma \in \left(\mathcal{S}/\mathcal{F}^i \right) \text{ if and only if } \left(z^i_\tau \right)_{\tau \in \mathcal{S}(\sigma)} = \left(x^i_\tau \right)_{\tau \in \mathcal{S}(\sigma)}.$$

For every sufficiently small $1 > \lambda > 0$, the allocation $x + \lambda(z - x)$ lies in $X(e)$ and Pareto dominates allocation x in $X(e)$ by strict convexity of preferences. In particular, by strict convexity of preferences, for every individual i in $\mathcal{J}$, at every date-event σ in $\mathcal{F}^i$,

$$(**) \qquad U_\sigma^i\left(x^i + \lambda\left(z^i - x^i\right)\right) > U_\sigma^i\left(e^i\right).$$

For every sufficiently large t in $\mathcal{T}$, the allocation $x^t + \lambda(z - x)$ lies in $X(e)$. Indeed, balancedness follows by construction; participation constraints are insured by continuity, because of (**), at all date-events σ in $\mathcal{F}^i$, and trivially, at all date-events σ in $\left(\mathcal{S}/\mathcal{F}^i\right)$. Finally, as it can assumed that

$$\bigcup_{i\in\mathcal{J}} \mathcal{F}^i \subset \mathcal{S}^t,$$

additional restrictions (†) are satisfied in every t-truncated planner problem along the sequence for every sufficiently large t in $\mathcal{T}$. This yields a contradiction. □

The sequence of steps proves the proposition. □

Proof of proposition 2. Using lemma 2, Malinvaud efficiency follows from the simple first-order characterization of equilibrium that is provided by Alvarez and Jermann [3, Propositions 4.5-4.6]. □

Proof of proposition 3. The proof is rather involved, so that we decompose it in several steps.

Recovering financial plans. To simplify notation, we introduce the positive linear operator $T : L \to L$ that is defined, at every date-event σ in $\mathcal{S}$, by

$$T(v)_\sigma = \frac{1}{p_\sigma} \sum_{\tau\in\sigma_+} p_\tau v_\tau.$$

For an individual i in $\mathcal{J}$, let g^i in L be given, at every date-event σ in $\mathcal{S}$, by

$$g_\sigma^i = \left(\frac{\pi_\sigma^i}{p_\sigma^i}\right)\left(U_\sigma^i\left(x^i\right) - U_\sigma^i\left(e^i\right)\right).$$

Notice that, by uniform impatience and feasibility, g^i is a bounded and positive element of L. Furthermore, observe that, by convexity of preferences, at every date-event σ in $\mathcal{S}$,

$$\begin{aligned}
\pi_\sigma^i\left(U_\sigma^i\left(x^i\right) - U_\sigma^i\left(e^i\right)\right) &\geq \pi_\sigma^i\left(u^i\left(x_\sigma^i\right) - u^i\left(e_\sigma^i\right)\right) + \sum_{\tau\in\sigma_+} \pi_\tau^i\left(U_\tau^i\left(x^i\right) - U_\tau^i\left(e^i\right)\right) \\
&\geq p_\sigma^i\left(x_\sigma^i - e_\sigma^i\right) + \sum_{\tau\in\sigma_+} \pi_\tau^i\left(U_\tau^i\left(x^i\right) - U_\tau^i\left(e^i\right)\right).
\end{aligned}$$

This, exploiting first-order conditions at a Malinvaud optimum, yields

$$T\left(g^i\right) + \left(x^i - e^i\right) \leq g^i.$$

Finally, define $g = \sum_{i\in\mathcal{J}} g^i$ and observe that g is a positive bounded element of L.

Define H as the set of all h in $[0, g]^{\mathcal{J}}$ satisfying

$$\sum_{i\in\mathcal{J}} h^i = g.$$

The set H is non-empty, convex and compact (in the product topology). Define a correspondence $f : H \to H$ by means of

$$f(h)_\sigma = \arg\min_{\hat{h}_\sigma \in H_\sigma} \sum_{i \in \mathcal{J}} \hat{h}^i_\sigma \left(T\left(g^i - h^i\right)_\sigma + \left(x^i - e^i\right)_\sigma - \left(g^i - h^i\right)_\sigma \right).$$

Basically, if a financial plan lies in the interior of the budget constraint at some date-event, current debt is increased. By construction, given any h in H, at every date-event σ in $\mathcal{S}$, there exists an individual i in $\mathcal{J}$ such that

$$T\left(g^i - h^i\right)_\sigma + \left(x^i - e^i\right)_\sigma \leq \left(g^i - h^i\right)_\sigma,$$

as

$$\sum_{i \in \mathcal{J}} \left(T\left(g^i - h^i\right) + \left(x^i - e^i\right) \right) = 0 = \sum_{i \in \mathcal{J}} \left(g^i - h^i\right).$$

As the correspondence $f : H \to H$ is closed with non-empty convex values, by Kakutani Fixed Point Theorem, it admits a fixed point h in H. At a fixed point, for every individual i in $\mathcal{J}$, at any date-event σ in $\mathcal{S}$,

$$T\left(g^i - h^i\right)_\sigma + \left(x^i - e^i\right)_\sigma > \left(g^i - h^i\right)_\sigma \text{ implies } h^i_\sigma = 0.$$

Hence,

$$g^i_\sigma \geq T\left(g^i\right)_\sigma + \left(x^i - e^i\right)_\sigma \geq T\left(g^i - h^i\right)_\sigma + \left(x^i - e^i\right)_\sigma > g^i_\sigma,$$

which is a contradiction. Thus, at a fixed point, for every individual i in $\mathcal{J}$,

$$T\left(g^i - h^i\right) + \left(x^i - e^i\right) \leq \left(g^i - h^i\right).$$

This suffices to prove budget-feasibility, as, for every individual i in $\mathcal{J}$,

$$T\left(g^i - h^i\right) + \left(x^i - e^i\right) = \left(g^i - h^i\right).$$

To conclude, for every individual i in $\mathcal{J}$, the financial plan $v^i = g^i - h^i$ in V^i is bounded, balances budget sequentially and satisfies, at every date-event σ in $\mathcal{S}$,

$$U^i_\sigma\left(x^i\right) = U^i_\sigma\left(e^i\right) \text{ only if } v^i_\sigma = g^i_\sigma - h^i_\sigma \leq g^i_\sigma \leq 0.$$

Furthermore, across individuals, financial plans v in V satisfy market clearing, that is,

$$\sum_{i \in \mathcal{J}} v^i = 0.$$

We treat such financial plans as given in the remaining parts of this proof. □

Individual optimality. For every individual i in $\mathcal{J}$, consider the set of all date-events at which this individual is at the autarchic utility, that is,

$$\mathcal{S}^i = \left\{ \sigma \in \mathcal{S} : U^i_\sigma\left(x^i\right) = U^i_\sigma\left(e^i\right) \right\}.$$

Also, define the space $F^i\left(x^i, v^i\right)$ of all debt limits f^i in F^i satisfying, at every date-event σ in $\mathcal{S}$,

$$v^i_\sigma + f^i_\sigma \geq 0$$

and, at every date-event σ in $\mathcal{S}^i$,

$$v^i_\sigma + f^i_\sigma = 0.$$

We here show that consumption plan x^i in X^i is optimal, subject to budget and debt constraints, given initial claims, at all debt limits f^i in $F^i\left(x^i, v^i\right)$.

Peg any date-event σ in $\mathcal{S}$. Observe that, as budget is balanced,

$$\frac{1}{p_\sigma}\sum_{\tau\in\sigma_+} p_\tau f^i_\tau - \frac{1}{p_\sigma}\sum_{\tau\in\sigma_+} p_\tau\left(v^i_\tau + f^i_\tau\right) - \left(x^i_\sigma - e^i_\sigma\right) \leq -v^i_\sigma.$$

Furthermore, considering any alternative budget feasible consumption plan z^i in X^i satisfying debt constraints,

$$-\frac{1}{p_\sigma}\sum_{\tau\in\sigma_+} p_\tau f^i_\tau + \frac{1}{p_\sigma}\sum_{\tau\in\sigma_+} p_\tau\left(w^i_\tau + f^i_\tau\right) + \left(z^i_\sigma - e^i_\sigma\right) \leq w^i_\sigma.$$

Using first-order conditions, one obtains

$$\frac{1}{p_\sigma}\sum_{\tau\in\sigma_+} p_\tau f^i_\tau - \frac{1}{p^i_\sigma}\sum_{\tau\in\sigma_+} p^i_\tau\left(v^i_\tau + f^i_\tau\right) - \left(x^i_\sigma - e^i_\sigma\right) \leq -v^i_\sigma$$

and

$$-\frac{1}{p_\sigma}\sum_{\tau\in\sigma_+} p_\tau f^i_\tau + \frac{1}{p^i_\sigma}\sum_{\tau\in\sigma_+} p^i_\tau\left(w^i_\tau + f^i_\tau\right) + \left(z^i_\sigma - e^i_\sigma\right) \leq w^i_\sigma.$$

Therefore, adding up terms, it follows that

$$(\ddagger) \qquad \sum_{\tau\in\sigma_+} p^i_\tau\left(w^i_\tau - v^i_\tau\right) + p^i_\sigma\left(z^i_\sigma - x^i_\sigma\right) \leq p^i_\sigma\left(w^i_\sigma - v^i_\sigma\right).$$

For every t in $\mathcal{T}$, let

$$\mathcal{S}_t = \{\sigma\in\mathcal{S} : t(\sigma) = t\} \text{ and } \mathcal{S}^t = \{\sigma\in\mathcal{S} : t(\sigma)\leq t\}.$$

Consolidating inequalities (‡) up to period t in $\mathcal{T}$, and using the fact the initial claims are given,

$$\sum_{\sigma\in\mathcal{S}^t} p^i_\sigma\left(z^i_\sigma - x^i_\sigma\right) \leq \sum_{\tau\in\mathcal{S}_{t+1}} p^i_\tau\left(v^i_\tau - w^i_\tau\right) \leq \sum_{\tau\in\mathcal{S}_{t+1}} p^i_\tau\left(v^i_\tau + f^i_\tau\right),$$

where the last inequality follows from debt constraints. By concavity of utility, this suffices to prove optimality, as the right hand-side vanishes in the limit, because p^i in P^i is a summable element of L and $v^i + f^i$ is a bounded element of L. □

Recovering debt limits. Given debt limits f^i in F^i, at every date-event $\bar\sigma$ in $\mathcal{S}$, let $B^i_{\bar\sigma}\left(w^i_{\bar\sigma}; f^i\right)$ be the set of all plans $\left(\bar x^i, \bar v^i\right)$ in $X^i\times V^i$ satisfying, at every date-event σ in $\mathcal{S}(\bar\sigma)$, budget constraint,

$$\sum_{\tau\in\sigma_+} p_\tau \bar v^i_\tau + p_\sigma\left(\bar x^i_\sigma - e^i_\sigma\right) \leq \bar v^i_\sigma,$$

and debt constraints,

$$-\left(\bar v^i_\tau + f^i_\tau\right)_{\tau\in\sigma_+} \leq 0,$$

given initial wealth $w^i_{\bar\sigma}$ in $\mathbb{R}$.

Consider the set

$$D^i = \left\{\left(w^i, f^i\right) : B^i_{\bar\sigma}\left(w^i_{\bar\sigma}; f^i\right) \text{ is non-empty at every } \bar\sigma\in\mathcal{S}\right\} \subset V^i\times F^i.$$

This domain is non-empty, closed and convex. Define a value function $J^i : D^i \to L$ by means of

$$J^i_{\bar\sigma}\left(w^i_{\bar\sigma}; f^i\right) = \max\left\{U^i_{\bar\sigma}\left(\bar x^i\right) : \left(\bar x^i, \bar v^i\right)\in B^i_{\bar\sigma}\left(w^i_{\bar\sigma}; f^i\right)\right\};$$

It is straightforward to verify that this value function is well-defined, as the maximum is achieved, and fulfils the following properties: (i) it is bounded; (ii) it is concave; (iii) it is weakly increasing in f^i in F^i and strictly increasing in w^i in V^i on its domain D^i; (iv) for every $\bar{f}^i$ in F^i, it is continuous on the restricted domain

$$\left\{\left(w^i, f^i\right) \in D^i : f^i = \bar{f}^i\right\}$$

and upper hemicontinuous on the restricted domain

$$\left\{\left(w^i, f^i\right\} \in D^i : f^i \leq \bar{f}^i\right\};$$

(v) for every f^i in $F^i\left(x^i, v^i\right)$, by construction, $\left(v^i, f^i\right)$ is an element of the domain D^i and, by the previous argument for optimality, at every date-event $\bar{\sigma}$ in $\mathcal{S}$,

$$J^i_{\bar{\sigma}}\left(v^i_{\bar{\sigma}}; f^i\right) = U^i_{\bar{\sigma}}\left(x^i\right).$$

We now show some properties of differentiability of the value function. Given any f^i in $F^i\left(x^i, v^i\right)$ and any w^i in V^i satisfying $w^i \geq v^i - x^i$, $\left(w^i, f^i\right)$ is an element of the domain D^i and, by optimality, at every date-event σ in $\mathcal{S}$,

$$\begin{aligned} J^i_\sigma\left(w^i_\sigma; f^i\right) &\geq u^i\left(x^i_\sigma + \left(w^i_\sigma - v^i_\sigma\right)\right) + \frac{1}{\pi^i_\sigma}\sum_{\tau\in\sigma_+}\pi^i_\tau J^i_\tau\left(v^i_\tau; f^i\right) \\ &\geq u^i\left(x^i_\sigma + \left(w^i_\sigma - v^i_\sigma\right)\right) + \frac{1}{\pi^i_\sigma}\sum_{\tau\in\sigma_+}\pi^i_\tau U^i_\tau\left(x^i\right). \end{aligned}$$

By the well-known result in convex analysis, given any f^i in $F^i\left(x^i, v^i\right)$, the value function admits a (partial) derivative at $\left(v^i, f^i\right)$ in D^i and, at every date-event σ in $\mathcal{S}$,

$$\partial J^i_\sigma\left(v^i_\sigma; f^i\right) = \partial u^i\left(x^i_\sigma\right).$$

Thus, given any f^i in $F^i\left(x^i, v^i\right)$, consider any $\left(w^i, f^i\right)$ in D^i satisfying, at some date-event σ in $\mathcal{S}$, $J^i_\sigma\left(w^i_\sigma; f^i\right) \geq U^i_\sigma\left(e^i\right)$ and $w^i_\sigma \leq v^i_\sigma$. Concavity delivers

$$\begin{aligned} U^i_\sigma\left(e^i\right) - U^i_\sigma\left(x^i\right) &\leq \\ J_\sigma\left(w^i_\sigma; f^i\right) - J_\sigma\left(v^i_\sigma; f^i\right) &\leq \partial u^i\left(x^i\right)\left(w^i_\sigma - v^i_\sigma\right) \\ &\leq \xi\left(w^i_\sigma - v^i_\sigma\right), \end{aligned}$$

where

$$\xi = \bigwedge_{\sigma\in\mathcal{S}} \partial u^i\left(x^i_\sigma\right) > 0.$$

Thus, rearranging terms,

$$v^i_\sigma - \frac{U^i_\sigma\left(x^i\right) - U^i_\sigma\left(e^i\right)}{\xi} \leq w^i_\sigma.$$

Also, by uniform impatience and boundedness of per-period utility, there exists a sufficiently large $\phi > 0$ satisfying

$$\phi > \bigvee_{\sigma\in\mathcal{S}} \frac{U^i_\sigma\left(x^i\right) - U^i_\sigma\left(e^i\right)}{\xi}.$$

It follows that, given any f^i in $F^i\left(x^i, v^i\right)$, for every $\left(w^i, f^i\right)$ in D^i,

$$(\ddagger) \qquad J^i_\sigma\left(w^i_\sigma; f^i\right) \geq U^i_\sigma\left(e^i\right) \text{ only if } w^i_\sigma \geq v^i_\sigma - \phi.$$

We shall exploit this fundamental inequality to recover debt limits.

We define an implicit operator $G^i : F^i\left(x^i, v^i\right) \to F^i\left(x^i, v^i\right)$ by setting, at every date-event σ in $\mathcal{S}$,

$$J^i_\sigma\left(-G^i\left(f^i\right)_\sigma; f^i\right) = U^i_\sigma\left(e^i\right).$$

To prove that this operator is well defined, pegging any date-event σ in $\mathcal{S}$, observe that

$$J^i_\sigma\left(0; f^i\right) \geq J^i_\sigma\left(0; 0\right) \geq U^i_\sigma\left(e^i\right)$$

and

$$J^i_\sigma\left(-g^i_\sigma; f^i\right) \leq u^i\left(0\right) + \frac{1}{\pi^i_\sigma}\sum_{\tau\in\sigma_+}\pi^i_\tau U^i_\tau\left(x^i\right) < U^i_\sigma\left(e^i\right),$$

where

$$0 \leq g^i_\sigma = \sup\left\{-w^i_\sigma \in \mathbb{R} : B^i_\sigma\left(w^i_\sigma; f^i\right) \text{ is non-empty}\right\} \leq e^i_\sigma + \frac{1}{p_\sigma}\sum_{\tau\in\sigma_+} p_\tau f^i_\tau,$$

as utility satisfies (weak) Inada conditions. Hence, by the Intermediate Value Theorem, $G^i\left(f^i\right)$ exists in L. Also, it is positive and bounded, as first-order conditions imply

$$\begin{aligned}
\frac{1}{p_\sigma}\sum_{\tau\in\sigma_+} p_\tau f^i_\tau &\leq \frac{1}{p_\sigma}\sum_{\tau\in\sigma_+} p_\tau \sum_{j\in\mathcal{J}} f^j_\tau \\
&= \frac{1}{p_\sigma}\sum_{\tau\in\sigma_+} p_\tau \sum_{j\in\mathcal{J}} \left(v^j_\tau + f^j_\tau\right) \\
&= \sum_{j\in\mathcal{J}} \frac{1}{p_\sigma}\sum_{\tau\in\sigma_+} p_\tau \left(v^j_\tau + f^j_\tau\right) \\
&= \sum_{j\in\mathcal{J}} \frac{1}{p^j_\sigma}\sum_{\tau\in\sigma_+} p^j_\tau \left(v^j_\tau + f^j_\tau\right) \\
&\leq \sum_{j\in\mathcal{J}} \left(\bigvee_{\tau\in\mathcal{S}} \left|v^j_\tau + f^j_\tau\right|\right) \frac{1}{p^j_\sigma}\sum_{\tau\in\sigma_+} p^j_\tau
\end{aligned}$$

and uniform impatience yields

$$\frac{1}{p^j_\sigma}\sum_{\tau\in\sigma_+} p^j_\tau \leq \left(\partial u^j\left(\epsilon\right) \Big/ \partial u^j\left(\frac{1}{\epsilon}\right)\right)\frac{1}{\pi^j_\sigma}\sum_{\tau\in\sigma_+}\pi^j_\tau \leq \eta\left(\partial u^j\left(\epsilon\right) \Big/ \partial u^j\left(\frac{1}{\epsilon}\right)\right).$$

Finally, at every date-event σ in $\mathcal{S}$,

$$J^i_\sigma\left(v^i_\sigma; f^i\right) = U^i_\sigma\left(x^i\right) \geq U^i_\sigma\left(e^i\right) = J^i_\sigma\left(-G^i\left(f^i\right)_\sigma; f^i\right)$$

and, at every date-event σ in $\mathcal{S}^i$,

$$J^i_\sigma\left(v^i_\sigma; f^i\right) = U^i_\sigma\left(x^i\right) = U^i_\sigma\left(e^i\right) = J^i_\sigma\left(-G^i\left(f^i\right)_\sigma; f^i\right).$$

Hence, $G^i\left(f^i\right)$ is an element of $F^i\left(x^i, v^i\right)$. Finally, observe that the operator $G^i : F^i\left(x^i, v^i\right) \rightarrow F^i\left(x^i, v^i\right)$ is (weakly) monotone.

Construct debt limits $\bar{f}^i$ in $F^i\left(x^i, v^i\right)$ so that

$$\bar{f}^i_\sigma = -v^i_\sigma, \text{ at every date-event } \sigma \in \mathcal{S}^i,$$

and

$$\bar{f}^i_\sigma \geq -v^i_\sigma + \phi, \text{ at every date-event } \sigma \in \left(\mathcal{S}/\mathcal{S}^i\right).$$

We claim that $G^i\left(\bar{f}^i\right)$ in $F^i\left(x^i, v^i\right)$ satisfies $G^i\left(\bar{f}^i\right) \leq \bar{f}^i$. Indeed, exploiting restriction (‡), at every date-event σ in $\left(\mathcal{S}/\mathcal{S}^i\right)$,

$$G^i\left(\bar{f}^i\right)_\sigma \leq -v^i_\sigma + \phi \leq \bar{f}^i_\sigma;$$

at every date-event σ in $\mathcal{S}^i$,

$$G^i\left(\bar{f}^i\right)_\sigma = -v^i_\sigma = \bar{f}^i_\sigma.$$

Now, by induction, construct a sequence $\left(\left(G^i\right)^n\left(\bar{f}^i\right)\right)_{n\in\mathcal{T}}$ in $F^i\left(x^i, v^i\right)$. Such a sequence is weakly decreasing and bounded, as

$$\bar{f}^i \geq \left(G^i\right)^n\left(\bar{f}^i\right) \geq \left(G^i\right)^{n+1}\left(\bar{f}^i\right) \geq -v^i.$$

Hence, it converges to some f^i in $F^i\left(x^i, v^i\right)$ in the product topology. By upper hemicontinuity of the value function, at every date-event σ in $\mathcal{S}$,

$$J^i_\sigma\left(-f^i_\sigma; f^i\right) \geq U^i_\sigma\left(e^i\right).$$

Suppose that there exists $\epsilon > 0$ such that, at some date-event σ in $\mathcal{S}$,

$$J_\sigma\left(-f^i_\sigma; f^i\right) > J^i_\sigma\left(-f^i_\sigma - \epsilon; f^i\right) > U^i_\sigma\left(e^i\right).$$

For every sufficiently large n in $\mathcal{T}$, $\left(G^i\right)^{n+1}\left(\bar{f}^i\right)_\sigma \leq f^i_\sigma + \epsilon$ and, therefore,

$$\begin{aligned} U^i_\sigma\left(e^i\right) &\geq J^i_\sigma\left(-\left(G^i\right)^{n+1}\left(\bar{f}^i\right)_\sigma; \left(G^i\right)^n\left(\bar{f}^i\right)\right) \\ &\geq J^i_\sigma\left(-f^i_\sigma - \epsilon; \left(G^i\right)^n\left(\bar{f}^i\right)\right) \\ &\geq J^i_\sigma\left(-f^i_\sigma - \epsilon; f^i\right) \\ &> U^i_\sigma\left(e^i\right), \end{aligned}$$

a contradiction. Hence, f^i in F^i are not-too-tight debt limits at equilibrium. □

The proof is now complete. □

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