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Solution to Problem 1008.

Proof. We know $f(x) = x^4 - 10x^2 + 1 \neq (x+a)(x^3 + bx^2 + cx + d)$ for any integers a, b, c and d.

Now suppose $f(x) = x^4 - 10x^2 + 1 = (x^2 + ax + b)(x^2 + cx + d) = x^4 + (a+c)x^3 + (b + ac + d)x^2 + (ad + bc)x + bd$. This implies $a + c = 0$, $b + ac + d = -10$, $ad + bc = 0$ and $bd = 1$.

Suppose $a = 0$ or $c = 0$. Then $b + d = -10$ and $bd = 1$. From the first of the previous two equations we have $b^2 + bd = -10b$, or $b^2 + 10b + 1 = 0$, which has no rational roots. Therefore, $a \neq 0$ or $c \neq 0$.

Suppose $a = -c$ then from $ad + bc = 0$, we have $b = d$. Therefore $b = d = \pm 1$ because $bd = 1$. If we try to solve for a or c, it follows that $a^2 = c^2 = 12$ or $a^2 = c^2 = 8$. Of course there is no solution in $\mathbb{Q}$.

Therefore $f(x) = x^4 - 10x^2 + 1$ is irreducible in $\mathbb{Q}$.

Lindsay Childs provided a proof to the following theorem in problem E2578 in *The American Mathematical Monthly*; Vol. 84, No. 5; May, 1977, pp. 390-1.

Theorem: Given any prime p and integers a and b, the polynomial $P(x) = x^4 + ax^2 + b^2$ is reducible mod p.

In our problem, $a = -10$ and $b^2 = 1$

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