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Solution to Problem 1010.

We will use a known series

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)!} = 1 \tag{1}$$

and the inequality

$$\frac{\sqrt{2\pi n}}{n!} < \frac{e^n}{n^n} < \frac{e^{\frac{1}{12n}}\sqrt{2\pi n}}{n!} \tag{2}$$

to look for a upper bound for the series

$$\sum_{n=0}^{\infty} \frac{e^n}{(n+1)^{n+1}}. \tag{3}$$

If we divide (2) by e, we can transform that into

$$\frac{\sqrt{2\pi(n+1)}}{e(n+1)!} < \frac{e^n}{(n+1)^{(n+1)}} < \frac{e^{\frac{1}{12(n+1)}}\sqrt{2\pi(n+1)}}{e(n+1)!}$$

or

$$\frac{\sqrt{2\pi}}{e}\sum_{n=0}^{\infty} \frac{\sqrt{(n+1)}}{(n+1)!} < \sum_{n=0}^{\infty} \frac{e^n}{(n+1)^{(n+1)}} < \frac{\sqrt{2\pi}}{e}\sum_{n=0}^{\infty} \frac{e^{\frac{1}{12(n+1)}}\sqrt{(n+1)}}{(n+1)!}. \tag{4}$$

A lower bound for (3) would be $\displaystyle\sum_{n=0}^{4} \frac{e^n}{(n+1)^{n+1}} \approx 2.03169833$. Since $e^{\frac{1}{12(n+1)}} < e^{\frac{1}{12}}$ for $n > 1$, we have

$$\frac{\sqrt{2\pi}}{e}\sum_{n=0}^{\infty}\frac{e^{\frac{1}{12(n+1)}}\sqrt{(n+1)}}{(n+1)!} < \frac{e^{\frac{1}{12}}\sqrt{2\pi}}{e}\sum_{n=0}^{\infty}\frac{\sqrt{(n+1)}}{(n+1)!}$$

Finally, we have the following:

$$\sum_{n=0}^{4}\frac{e^n}{(n+1)^{n+1}} < \sum_{n=0}^{\infty}\frac{e^n}{(n+1)^{(n+1)}} < \frac{e^{\frac{1}{12}}\sqrt{2\pi}}{e}\sum_{n=0}^{\infty}\frac{\sqrt{(n+1)}}{(n+1)!}.$$

Notice that $\displaystyle\sum_{n=2}^{\infty}\frac{\sqrt{(n+1)}}{(n+1)!} < \sum_{n=2}^{\infty}\frac{n}{(n+1)!} = \frac{1}{2}$. Therefore

$$\frac{e^{\frac{1}{12}}\sqrt{2\pi}}{e}\sum_{n=0}^{\infty}\frac{\sqrt{(n+1)}}{(n+1)!} < \frac{e^{\frac{1}{12}}\sqrt{2\pi}}{e}\left(\frac{\sqrt{1}}{1!}+\frac{\sqrt{2}}{2!}+\frac{1}{2}\right) \approx 2.212126733.$$

So we have a lower and upper bound,

$$2.03169833 < \sum_{n=0}^{\infty}\frac{e^n}{(n+1)^{(n+1)}} < 2.212126733$$

From here, it is easy to verify that

$$\left|\sum_{n=0}^{\infty}\frac{e^n}{(n+1)^{(n+1)}} + \frac{1}{2} - e\right| < \frac{1}{2}.$$

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