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Solution to Problem 1012.

I will reserve the letter p for primes. Let's say n is a number such that the product of its proper divisors is n. Assume $n = p_1 p_2 \cdots p_m$. Then the proper divisors of n are $\{1\}$, $\{p_i \mid p_1, p_2, \ldots, p_m\}$, $\{p_i p_j \mid p_1 p_2,\ p_1 p_3, \ldots, p_1 p_m,$ $p_2 p_3,\ p_2 p_3, \ldots, \}$, $\{p_i p_j p_k | p_1 p_2 p_3,\ p_1 p_2 p_4, \ldots, p_1 p_2 p_m,\ p_2 p_3 p_4,\ p_2 p_3 p_5, \ldots, \}, \ldots$ i.e. sets of factors taking 1 prime factor at a time, 2 prime factors at a time, 3 prime factors at a time and up to $(m-1)$ prime factors at a time.

Obviously, if $m \geq 3$ the product of the proper divisors of n cannot be n itself, less to say $n = p_1^{e_1} p_2^{e_2} \cdots p_m^{e_m}$.

This is true only if $n = p_1 p_2$ with $p_1 \neq p_2$ or $n = p^3$.

Armed with this, suppose n is an even perfect number, then $n = 2^{k-1}(2^k - 1)$ where $(2^k - 1)$ is prime. The only perfect number that meets the above criteria is 6.

Notice that the existence of an odd perfect number does not affect our result since Sylvester proved that an odd perfect number must contain at least 8 distinct prime factors.

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