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Solution to Problem 1016.

Suppose the distance from the center of a regular n-gon to a vertex is $d_0 = 1$. Then the length of a side of the n-gon is $s_0 = 2\sin(\frac{\pi}{n})$ and the first inscribed circle has a radius $r_0 = \cos(\frac{\pi}{n})$. In the second inscribed n-gon, the distance from the center to a vertex is $d_1 = \cos(\frac{\pi}{n})$ and has a side $s_1 = 2\sin(\frac{\pi}{n})\cos(\frac{\pi}{n})$. The radius of the circle inscribed in the second n-gon is $r_1 = \cos^2(\frac{\pi}{n})$. In general in the n^{th} n-gon, $d_{n-1} = \cos^{n-1}(\frac{\pi}{n})$, $s_{n-1} = 2\sin(\frac{\pi}{n})\cos^{n-1}(\frac{\pi}{n})$ and $r_{n-1} = \cos^n(\frac{\pi}{n})$.

If we just look at the triangle formed by the lines from the center to any two adjacent vertices of the first n-gon and the side between these two vertices, then the shaded area

$$\begin{aligned}
A_s(n) &= \sum_{i=0}^{\infty}\left[\frac{1}{n}\pi r_i^2 - \frac{1}{2}s_{i+1}r_{i+1}\right] \\
&= \sum_{i=0}^{\infty}\left[\frac{1}{n}\pi\cos^{2(i+1)}\left(\frac{\pi}{n}\right) - \sin\left(\frac{\pi}{n}\right)\cos^{i+1}\left(\frac{\pi}{n}\right)\cos^{i+2}\left(\frac{\pi}{n}\right)\right] \\
&= \left[\frac{1}{n}\pi\cos^2\left(\frac{\pi}{n}\right) - \sin\left(\frac{\pi}{n}\right)\cos^3\left(\frac{\pi}{n}\right)\right]\sum_{i=0}^{\infty}\cos^{2i}\left(\frac{\pi}{n}\right) \\
&= \left[\frac{1}{n}\pi\cos^2\left(\frac{\pi}{n}\right) - \sin\left(\frac{\pi}{n}\right)\cos^3\left(\frac{\pi}{n}\right)\right]\left(\frac{1}{1-\cos^2\left(\frac{\pi}{n}\right)}\right) \\
&= \frac{1}{\sin^2\left(\frac{\pi}{n}\right)}\left[\frac{1}{n}\pi\cos^2\left(\frac{\pi}{n}\right) - \sin\left(\frac{\pi}{n}\right)\cos^3\left(\frac{\pi}{n}\right)\right]
\end{aligned}$$

And the area of the corresponding sector of the first inscribed circle is $A_0(n) = \frac{1}{n}\pi r_0^2 = \frac{1}{n}\pi\cos^2\left(\frac{\pi}{n}\right)$. Therefore the desired ratio is

$$\frac{A_s(n)}{A_0(n)} = \frac{\frac{1}{\sin^2\left(\frac{\pi}{n}\right)}\left[\frac{1}{n}\pi\cos^2\left(\frac{\pi}{n}\right) - \sin\left(\frac{\pi}{n}\right)\cos^3\left(\frac{\pi}{n}\right)\right]}{\frac{1}{n}\pi\cos^2\left(\frac{\pi}{n}\right)}$$

$$= \frac{\frac{\pi}{n} - \sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}\sin^2\left(\frac{\pi}{n}\right)}$$

(a) $n = 3$

$$\begin{aligned}\frac{A_s(3)}{A_0(3)} &= \frac{\frac{\pi}{3} - \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{3}\right)}{\frac{\pi}{3}\sin^2\left(\frac{\pi}{3}\right)} \\ &= \frac{4}{3} - \frac{\sqrt{3}}{\pi}\end{aligned}$$

(b) limit $n \to \infty$

$$\lim_{n\to\infty} \frac{A_s(n)}{A_0(n)} = \lim_{n\to\infty} \frac{\frac{\pi}{n} - \sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}\sin^2\left(\frac{\pi}{n}\right)}$$

Let $f(x) = x - \sin x \cos x$ and $g(x) = x \sin^2 x$. Then the above limit becomes

$$\lim_{x\to 0} \frac{f(x)}{g(x)}$$

Apply l'Hôpital's rule three times gives

$$\begin{aligned}\lim_{x\to 0} \frac{f(x)}{g(x)} &\sim \lim_{x\to 0} \frac{f^{(3)}(x)}{g^{(3)}(x)} \\ &= \lim_{x\to 0} \frac{4\cos^2 x - 4\sin^2 x}{6\cos^2 x - 8x\cos x \sin x - 6\sin^2 x} \\ &= \frac{2}{3}\end{aligned}$$

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