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Solution to Problem 1019.

$$\frac{(1^7 - 3^7 + 5^7 - \cdots + (-1)^{n+1}(2n-1)^7) - 28(1^3 - 3^3 + 5^3 - \cdots + (-1)^{n+1}(2n-1)^3)}{1 - 3 + 5 - \cdots + (2n-1)}$$

Case (I) $n = 2k+1$ for $k = 0, 1, 2, \ldots$

Then the above expression can be rewritten as

$$\frac{\left(1 + \sum_{i=1}^{k} \left[-(4i-1)^7 + (4i+1)^7\right]\right) - 28\left(1 + \sum_{i=1}^{k} \left[-(4i-1)^3 + (4i+1)^3\right]\right)}{1 + \sum_{i=1}^{k} 2}$$

$$= \frac{-27 + \sum_{i=1}^{k} \left(57344i^6 + 17920i^4 - 2016i^2 - 54\right)}{2k+1}$$

$$= \frac{8192k^7 + 28672k^6 + 32256k^5 + 8960k^4 - 4256k^3 - 1008k^2 + 378k - 27}{2k+1}$$

$$= 4096k^6 + 12288k^5 + 9984k^4 - 512k^3 - 1872k^2 + 432k - 27$$

$$= (16k^2 + 16k - 3)^3$$

Case (II) $n = 2k$ for $k = 0, 1, 2, \ldots$

The original expression can be rewritten as

$$\frac{\left(1 + \sum_{i=1}^{k} \left[-(4i-1)^7 + (4i+1)^7\right] - (4k+3)^7\right) - 28\left(1 + \sum_{i=1}^{k} \left[-(4i-1)^3 + (4i+1)^3\right] - (4k+3)^3\right)}{1 + \sum_{i=1}^{k} 2 - [4(k+1) - 1]}$$

$$= \frac{-8192k^7 - 57344k^6 - 161280k^5 - 232960k^4 - 183904k^3 - 78624k^2 - 17010k - 1458}{-(2k+2)}$$

$$= 4096k^6 + 24576k^5 + 56064k^4 + 60416k^3 + 31536k^2 + 7776k + 729$$

$$= (16k^2 + 32k + 9)^3$$

This shows the original expression is a cube. For it to be a sixth power, then either $16k^2+16k-3$ or $16k^2+32k+9$ is a square. It is easy to see that $16k^2+16k-3$ can never be a square. $16k^2+32k+9$ is a square only if $k=0$.

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ADDENDUM

$$\begin{aligned}
\sum_{i=1}^{n} i &= \frac{n(n+1)}{2} \\
\sum_{i=1}^{n} i^2 &= \frac{n(n+1)(2n+1)}{6} \\
\sum_{i=1}^{n} i^3 &= \left[\frac{n(n+1)}{2}\right]^2 \\
\sum_{i=1}^{n} i^4 &= \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1) \\
\sum_{i=1}^{n} i^5 &= \frac{1}{12}n^2(n+1)^2(2n^2+2n-1) \\
\sum_{i=1}^{n} i^6 &= \frac{1}{42}n(n+1)(2n+1)(3n^4+6n^3-3n+1)
\end{aligned}$$