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Solution to Problem 1023.

$$U_1 = 16 \text{ and for } n \geq 1, U_{n+1} = U_n + 8n + 12.$$

$$U_1 = 16$$

$$U_2 = U_1 + 8 \times 1 + 12 = 16 + 8 \times 1 + 12$$

$$U_3 = U_2 + 8 \times 2 + 12 = 16 + 8 \times 1 + 12 + 8 \times 2 + 12$$

$$U_4 = U_2 + 8 \times 3 + 12 = 16 + 8 \times 1 + 12 + 8 \times 2 + 12 + 8 \times 3 + 12$$

\vdots

$$U_n = 16 + 8 \sum_{i=1}^{n-1} i + 12(n-1)$$

$$= 16 + 8 \times \frac{n(n-1)}{2} + 12(n-1)$$

$$= 4(n+1)^2$$

Then

$$\begin{aligned} \sum_{n=0}^{\infty} U_{n+1} &= \sum_{n=1}^{\infty} U_n \\ &= \sum_{n=1}^{\infty} \frac{1}{4(n+1)^2} \\ &= \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \\ &= \frac{1}{4} (\zeta(2) - 1) \\ &= \frac{1}{4} \left(\frac{\pi^2}{6} - 1 \right) \\ &= \frac{\pi^2}{24} - \frac{1}{4} \end{aligned}$$

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