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Solution to Problem 1025.

The answer is actually given a few pages later (p.280) in the solution to problem 998.

$$\sum_{k=0}^{n}(-1)^k\frac{\binom{n}{k}}{x+k}=\frac{n!\Gamma(x)}{\Gamma(x+n+1)},$$

where $\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt=(z-1)\Gamma(z-1)$ is the Euler Gamma function.

Let's define the difference function

$$\begin{aligned}\triangle f(x) &= f(x+1)-f(x)\\ \triangle^n f(x) &= \triangle^{n-1}f(x+1)-\triangle^{n-1}f(x),\ \text{for } n\geq 2.\end{aligned}$$

Then

$$\triangle^n f(x)=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f(x+k).$$

Let's define

$$f_m(x)=\begin{cases}\frac{\Gamma(x+1)}{\Gamma(m+1)}=x(x-1)(x-2)\cdots(x-m+1) & ,\ \text{for } m>0\\ 1 & ,\ \text{for } m=0.\\ \frac{\Gamma(x+1)}{\Gamma(x-m+1)}=\frac{1}{(x+1)(x+2)(x+3)\cdots(x+m)} & ,\ \text{for } m<0.\end{cases}$$

Again, if we apply the difference function defined as above to $f_m(x)$, we get

$$\begin{aligned}\triangle f_m(x) &= f_m(x+1)-f_m(x)=mf_{m-1}(x)\\ \triangle^2 f_m(x) &= \triangle f_m(x+1)-\triangle f_m(x)=m(m-1)f_{m-2}(x)\\ &\vdots\\ \triangle^n f_m(x) &= \triangle^{n-1}f_m(x+1)-\triangle^{n-1}f_m(x)\\ &= m(m-1)(m-2)\cdots(m-n+1)f_{m-n}(x)\end{aligned}$$

The function we are interested in is $f(x) = f_{-1}(x-1) = \frac{1}{x}$. In which case

$$\triangle^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+k) = \sum_{k=0}^{n} (-1)^{n-k} \frac{\binom{n}{k}}{x+k}$$

and

$$\begin{aligned} \triangle^n f_{-1}(x-1) &= (-1)^n 1 \cdot 2 \cdot 3 \cdots n \cdot f_{-(n+1)}(x-1) \\ &= \frac{(-1)^n n!}{x(x+1)(x+2)\cdots(x+n)} \\ &= \frac{(-1)^n n! \Gamma(x)}{\Gamma(x+n+1)}. \end{aligned}$$

Since $\triangle^n f(x) = \triangle^n f_{-1}(x-1)$, we have

$$\sum_{k=0}^{n} (-1)^{n-k} \frac{\binom{n}{k}}{x+k} = \frac{(-1)^n n! \Gamma(x)}{\Gamma(x+n+1)}.$$

Or simply

$$\sum_{k=0}^{n} (-1)^{k} \frac{\binom{n}{k}}{x+k} = \frac{n! \Gamma(x)}{\Gamma(x+n+1)}$$

where $x \notin \{0, -1, -2, -3, \ldots, -n\}$.

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