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Solution # 1 to Problem 1030.

That the three lines $\overline{AA_1}$, $\overline{BB_1}$ and $\overline{CC_1}$ are concurrent is well known. The point of concurrency is called the Fermat point or the Torricelli point of $\triangle ABC$. Also known is that $\|\overline{AA_1}\| = \|\overline{BB_1}\| = \|\overline{CC_1}\|$ (By rotating around point B by $\frac{\pi}{3}$ clockwise to map $\overline{C_1C}$ onto $\overline{AA_1}$, etc.). And if P is the point of concurrency, then $\angle APB_1 = \angle B_1PC = \angle CPA_1 = \angle A_1PB = \angle BPC_1 = \angle C_1PA = \frac{\pi}{3}$.

Now connect points A and A_2, B and B_2, and finally C and C_2. Rotate the figure around point A_1 clockwise by $\frac{\pi}{3}$. B_2 is mapped to C_1, B to C and B_1 to C_2. It is clear that $\overline{BB_2}$ is mapped onto $\overline{CC_1}$ and $\overline{BB_1}$ onto $\overline{CC_2}$. Therefore, $\|\overline{BB_2}\| = \|\overline{CC_1}\| = \|\overline{BB_1}\| = \|\overline{CC_2}\|$.

The implication of this rotation is that if we extend $\overline{B_2B}$ to intersect $\overline{CC_1}$ at point Q, then $\angle BQC_1 = \frac{\pi}{3}$. Since we know $\angle BPC_1 = \frac{\pi}{3}$, we conclude that points P and Q are the same and therefore lines $\overline{BQ}$ and $\overline{BP}$ are the same line. Therefore, points B_2, B and B_1 form a straight line. Likewise, points C_1, C and C_1 fall on a straight line.

By the same reasoning, if we rotate the figure around point B_1 clockwise by $\frac{\pi}{3}$, we have points A_1, A and A_1 forming a straight line.

One surprise is that points A, B and C are the midpoints of $\overline{A_1A_2}$, $\overline{B_1B_2}$ and $\overline{C_1C_2}$ respectively.

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Solution # 2

The problem can also be solved using vectors (with a lot of mechanical computation). Let the coordinates of point B be $(0,0)$, point A be (x_1, y_1) and point C be $(x_2, 0)$.

The rotation matrix $rot(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ rotates a vector counterclockwise by $\theta°$ or θ radian. The two we are using are $rot(\frac{\pi}{3}) = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ and $rot(-\frac{\pi}{3}) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$.

Then $\overrightarrow{BA} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$, $\overrightarrow{BC} = \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$ and $\overrightarrow{CA} = \begin{bmatrix} x_1 - x_2 \\ y_1 \end{bmatrix}$.

The coordinates of C_1 can be found be rotating $\overrightarrow{BA}$ counterclockwise $\frac{\pi}{3}$ radians. $\overrightarrow{BC_1} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \frac{x_1 - \sqrt{3}y_1}{2} \\ \frac{\sqrt{3}x_1 + y_1}{2} \end{bmatrix}$.

A_1 is $\overrightarrow{BC}$ rotated clockwise by $\frac{\pi}{3}$. $\overrightarrow{BA_1} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{x_2}{2} \\ \frac{-\sqrt{3}x_2}{2} \end{bmatrix}$.

And B_1 is a clockwise $\frac{\pi}{3}$ radian rotation of $\overrightarrow{CA}$ then a translation of $\overrightarrow{BC}$. $\overrightarrow{BB_1} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 - x_2 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{x_1 + x_2 + \sqrt{3}y_1}{2} \\ \frac{-\sqrt{3}x_1 + \sqrt{3}x_2 + y_1}{2} \end{bmatrix}$.

Similarly, the coordinates of C_2 is a clockwise rotation of $\overrightarrow{A_1B_1}$ by $\frac{\pi}{3}$ then a translation of $\overrightarrow{BA_1}$. $\overrightarrow{BC_2} = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \left(\begin{bmatrix} \frac{x_1 + x_2 + \sqrt{3}y_1}{2} \\ \frac{-\sqrt{3}x_1 + \sqrt{3}x_2 + y_1}{2} \end{bmatrix} - \begin{bmatrix} \frac{x_2}{2} \\ \frac{-\sqrt{3}x_2}{2} \end{bmatrix} \right) + \begin{bmatrix} \frac{x_2}{2} \\ \frac{-\sqrt{3}x_2}{2} \end{bmatrix} = \begin{bmatrix} \frac{-x_1 + 4x_2 + \sqrt{3}y_1}{2} \\ \frac{-\sqrt{3}x_1 - y_1}{2} \end{bmatrix}$.

B_2 is a counterclockwise rotation of $\frac{\pi}{3}$ of $\overrightarrow{A_1C_1}$ followed by a translation of $\overrightarrow{BA_1}$. $\overrightarrow{BB_2} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \left(\begin{bmatrix} \frac{x_1 - \sqrt{3}y_1}{2} \\ \frac{\sqrt{3}x_1 + y_1}{2} \end{bmatrix} - \begin{bmatrix} \frac{x_2}{2} \\ \frac{-\sqrt{3}x_2}{2} \end{bmatrix} \right) + \begin{bmatrix} \frac{x_2}{2} \\ \frac{-\sqrt{3}x_2}{2} \end{bmatrix} = \begin{bmatrix} \frac{-x_1 - x_2 - \sqrt{3}y_1}{2} \\ \frac{\sqrt{3}x_1 - \sqrt{3}x_2 - y_1}{2} \end{bmatrix}$.

And A_2 is a counterclockwise $\frac{\pi}{3}$ radian rotation of $\overrightarrow{C_1B_1}$ plus a translation of $\overrightarrow{BC_1}$. $\overrightarrow{BA_2} = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \left(\begin{bmatrix} \frac{x_1+x_2+\sqrt{3}y_1}{2} \\ \frac{-\sqrt{3}x_1+\sqrt{3}x_2+y_1}{2} \end{bmatrix} - \begin{bmatrix} \frac{x_1-\sqrt{3}y_1}{2} \\ \frac{\sqrt{3}x_1+y_1}{2} \end{bmatrix} \right) + \begin{bmatrix} \frac{x_1-\sqrt{3}y_1}{2} \\ \frac{\sqrt{3}x_1+y_1}{2} \end{bmatrix} = \begin{bmatrix} \frac{4x_1-x_2}{2} \\ \frac{\sqrt{3}x_2+4y_1}{2} \end{bmatrix}$.

Finally, we have all the coordinates

$$\begin{aligned}
&A(x_1, y_1)\\
&B(0,0)\\
&C(x_2,0)\\
&A_1(\tfrac{x_2}{2}, -\tfrac{\sqrt{3}x_2}{2})\\
&B_1(\tfrac{x_1+x_2+\sqrt{3}y_1}{2}, \tfrac{-\sqrt{3}x_1+\sqrt{3}x_2+y_1}{2})\\
&C_1(\tfrac{x_1-\sqrt{3}y_1}{2}, \tfrac{\sqrt{3}x_1+y_1}{2})\\
&A_2(\tfrac{4x_1-x_2}{2}, \tfrac{\sqrt{3}x_2+4y_1}{2})\\
&B_2(\tfrac{-x_1-x_2-\sqrt{3}y_1}{2}, \tfrac{\sqrt{3}x_1-\sqrt{3}x_2-y_1}{2})\\
&C_2(\tfrac{-x_1+4x_2+\sqrt{3}y_1}{2}, \tfrac{-\sqrt{3}x_1-y_1}{2})
\end{aligned}$$

Now it is a matter of arithmetics to see that $\overrightarrow{AA_1} = \begin{bmatrix} \frac{-2x_1+x_2}{2} \\ \frac{-\sqrt{3}x_2-2y_1}{2} \end{bmatrix}$, $\overrightarrow{AA_2} = \begin{bmatrix} \frac{2x_1-x_2}{2} \\ \frac{\sqrt{3}x_2+2y_1}{2} \end{bmatrix}$, $\overrightarrow{BB_1} = \begin{bmatrix} \frac{x_1+x_2+\sqrt{3}y_1}{2} \\ \frac{-\sqrt{3}x_1+\sqrt{3}x_2+y_1}{2} \end{bmatrix}$, $\overrightarrow{BB_2} = \begin{bmatrix} \frac{-x_1-x_2-\sqrt{3}y_1}{2} \\ \frac{\sqrt{3}x_1-\sqrt{3}x_2-y_1}{2} \end{bmatrix}$, $\overrightarrow{CC_1} = \begin{bmatrix} \frac{x_1-2x_2-\sqrt{3}y_1}{2} \\ \frac{\sqrt{3}x_1+y_1}{2} \end{bmatrix}$ and $\overrightarrow{CC_2} = \begin{bmatrix} \frac{-x_1+2x_2+\sqrt{3}y_1}{2} \\ \frac{-\sqrt{3}x_1-y_1}{2} \end{bmatrix}$.

The fact that $\overrightarrow{AA_1} = -\overrightarrow{AA_2}$, $\overrightarrow{BB_1} = -\overrightarrow{BB_2}$ and $\overrightarrow{CC_1} = -\overrightarrow{CC_2}$ shows each set of points $\{A, A_1, A_2\}$, $\{B, B_1, B_2\}$ and $\{C, C_1, C_2\}$ lie on a straight line.

Again, $\|\overrightarrow{AA_1}\| = \|\overrightarrow{AA_2}\| = \|\overrightarrow{BB_1}\| = \|\overrightarrow{BB_2}\| = \|\overrightarrow{CC_1}\| = \|\overrightarrow{CC_2}\|$ tell us that A is the midpoint of segment A_1A_2, B the midpoint of B_1B_2 and C the midpoint of C_1C_2.

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