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Solution to Problem 1034.

$a, b, c \in \mathbb{Z}^+$, show $\left(\frac{a+b+c}{3}\right)^{a+b+c} \geq \left(\frac{a+b}{2}\right)^c \left(\frac{b+c}{2}\right)^a \left(\frac{a+c}{2}\right)^b$.

Proof: The AM-GM inequality tells us that for positive real numbers x, y and z, $\frac{x+y+z}{3} \geq \sqrt[3]{xyz}$ with equality if and only if $x = y = z$. Let $x = a+b$, $y = b+c$ and $z = a+c$. We have

$$\begin{aligned}
\frac{(a+b)+(b+c)+(a+c)}{3} &\geq \sqrt[3]{(a+b)(b+c)(a+c)} \\
\frac{2(a+b+c)}{3} &\geq \sqrt[3]{(a+b)(b+c)(a+c)} \\
\left(\frac{2}{3}\right)^3 (a+b+c)^3 &\geq (a+b)(b+c)(a+c) \\
\frac{(a+b+c)^3}{(a+b)(b+c)(a+c)} &\geq \left(\frac{3}{2}\right)^3
\end{aligned}$$

If we take the logarithm of the last expression, we have

$$\ln\left(\frac{a+b+c}{a+b}\right) + \ln\left(\frac{a+b+c}{b+c}\right) + \ln\left(\frac{a+b+c}{a+c}\right) \geq 3\ln\left(\frac{3}{2}\right) \qquad (1)$$

Next, we will apply the principle of rearrangement inequality. Namely, for positive real numbers $x_1 \geq x_2 \geq x_3$ and $y_1 \geq y_2 \geq y_3$, if z_1, z_2, z_3 is any rearrangement (permutation) of y_1, y_2, y_3, then

$$x_1y_1 + x_2y_2 + x_3y_3 \geq x_1z_1 + x_2z_2 + x_3z_3 \geq x_1y_3 + x_2y_2 + x_3y_1.$$

Note that equality is achieved if and only if $x_1 = x_2 = x_3$ and $y_1 = y_2 = y_3$.

Using the above, we can obtain the following:

$$\begin{aligned}
x_1y_1 + x_2y_2 + x_3y_3 &= x_1y_1 + x_2y_2 + x_3y_3 \\
x_1y_1 + x_2y_2 + x_3y_3 &\geq x_1y_3 + x_2y_1 + x_3y_2 \\
x_1y_1 + x_2y_2 + x_3y_3 &\geq x_1y_2 + x_2y_3 + x_3y_1
\end{aligned}$$

Adding the above three expressions gives $3(x_1y_1+x_2y_2+x_3y_3)$ as the left hand side and $(x_1+x_2+x_3)(y_1+y_2+y_3)$ as the right hand side. Therefore,

$$3(x_1y_1+x_2y_2+x_3y_3) \geq (x_1+x_2+x_3)(y_1+y_2+y_3). \tag{2}$$

Getting back to our original problem, without loss of generality, let's assume $a \geq b \geq c$. It follows that $\frac{a+b+c}{b+c} \geq \frac{a+b+c}{a+c} \geq \frac{a+b+c}{a+b}$ and that $\ln\left(\frac{a+b+c}{b+c}\right) \geq \ln\left(\frac{a+b+c}{a+c}\right) \geq \ln\left(\frac{a+b+c}{a+b}\right)$.

Let $x_1 = a$, $x_2 = b$, $x_3 = c$, $y_1 = \ln\left(\frac{a+b+c}{b+c}\right)$, $y_2 = \ln\left(\frac{a+b+c}{a+c}\right)$ and $y_3 = \ln\left(\frac{a+b+c}{a+b}\right)$. Then we have

$$\begin{aligned}
3\left[a\ln\left(\frac{a+b+c}{b+c}\right)+b\ln\left(\frac{a+b+c}{a+c}\right)+c\ln\left(\frac{a+b+c}{a+b}\right)\right] &\geq \\
(a+b+c)\left[\ln\left(\frac{a+b+c}{b+c}\right)+\ln\left(\frac{a+b+c}{a+c}\right)+\ln\left(\frac{a+b+c}{a+b}\right)\right] &\geq \\
(a+b+c)\left[3\ln\left(\frac{3}{2}\right)\right]&
\end{aligned}$$

The first half of the inequality is a simple substitution from (2) while the second half from (1).

The key inequality is

$$a\ln\left(\frac{a+b+c}{b+c}\right)+b\ln\left(\frac{a+b+c}{a+c}\right)+c\ln\left(\frac{a+b+c}{a+b}\right) \geq (a+b+c)\ln\left(\frac{3}{2}\right)$$

which becomes

$$\begin{aligned}
\ln\left(\frac{a+b+c}{b+c}\right)^a+\ln\left(\frac{a+b+c}{a+c}\right)^b+\ln\left(\frac{a+b+c}{a+b}\right)^c &\geq \ln\left(\frac{3}{2}\right)^{a+b+c} \\
\ln\left[\left(\frac{a+b+c}{b+c}\right)^a\left(\frac{a+b+c}{a+c}\right)^b\left(\frac{a+b+c}{a+b}\right)^c\right] &\geq \ln\left(\frac{3}{2}\right)^{a+b+c} \\
\frac{(a+b+c)^{a+b+c}}{(a+b)^c(a+c)^b(b+c)^a} &\geq \left(\frac{3}{2}\right)^{a+b+c}
\end{aligned}$$

Finally a little manipulation shows

$$\left(\frac{a+b+c}{3}\right)^{a+b+c} \geq \left(\frac{a+b}{2}\right)^c\left(\frac{b+c}{2}\right)^a\left(\frac{a+c}{2}\right)^b$$

with equality only if $a = b = c$.

Note that we've been assuming $a, b, c \in \mathbb{R}^+$. Also known is $a^a b^b c^c \geq \left(\frac{a+b+c}{3}\right)^{a+b+c}$.

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