Rex H. Wu
Brooklyn, NY

Solution to Problem 1035.

Note that the binomial expansion of $(2+1)^{2n+1}$ is $\sum_{k=0}^{2n+1} 2^k \binom{2n+1}{k}$.

$$\begin{aligned}
\sum_{k=0}^{2n+1} 2^k \binom{2n+1}{k} &= 2^0 \binom{2n+1}{0} + \sum_{k=1}^{2n+1} 2^k \binom{2n+1}{k} \\
(2+1)^{2n+1} &= 1 + 2\sum_{k=1}^{2n+1} 2^{k-1} \binom{2n+1}{k} \\
3^{2n+1} &= 1 + 2x \\
x &= \frac{3^{2n+1}-1}{2}
\end{aligned}$$

It is readily seen that $\frac{1}{2}(x^2-1) \neq N(N+1)$. A counterexample is when $n=1$, we have $x=13$ and $\frac{1}{2}(13^2-1)=84$.

However it is true that $\frac{1}{4}(x^2-1) = N(N+1)$, for some integer N.

$$\begin{aligned}
\frac{1}{4}(x^2-1) &= \frac{1}{4}\left[\left(\frac{3^{2n+1}-1}{2}\right)^2 - 1\right] \\
&= \frac{1}{16}\left[\left(3^{2n+1}\right)^2 - 2\left(3^{2n+1}\right) - 3\right] \\
&= \left(\frac{3^{2n+1}+1}{4}\right)\left(\frac{3^{2n+1}-3}{4}\right)
\end{aligned}$$

From the last line, it is obvious that $\frac{3^{2n+1}+1}{4} - \frac{3^{2n+1}-3}{4} = 1$. It remains to show that anyone of them is an integer. Since $3 \equiv 3 \pmod 4$ and $3^{2n} \equiv 1 \pmod 4$, we have $3^{2n+1} \equiv 3 \pmod 4$. Or equivalently, $3^{2n+1} - 3 \equiv 0$ (mod 4)

■