Rex H. Wu
Brooklyn, NY

Solution to Problem 1037.

Should it be $\lim\limits_{n\to\infty}$ rather than $\lim\limits_{x\to\infty}$?

Observe that $\lfloor nx \rfloor = i$ for some integer i and $x \in [\frac{i}{n}, \frac{i+1}{n})$, where (a,b) denotes $a < x < b$ and $[a,b]$ denotes $a \le x \le b$.

First, the interval [0,1] is divided into n sub-intervals of length $\frac{1}{n}$. Then the integrand is integrated over each sub-interval.

$$\begin{aligned}
\lim_{n\to\infty} \int_0^1 \left(\frac{\lfloor nx \rfloor}{n}\right)^2 dx &= \lim_{n\to\infty} \sum_{i=0}^{n-1} \int_{\frac{i}{n}}^{\frac{i+1}{n}} \left(\frac{\lfloor nx \rfloor}{n}\right)^2 dx \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} \int_{\frac{i}{n}}^{\frac{i+1}{n}} \left(\frac{i}{n}\right)^2 dx \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} \left(\frac{i}{n}\right)^2 x\Big|_{\frac{i}{n}}^{\frac{i+1}{n}} \\
&= \lim_{n\to\infty} \sum_{i=0}^{n-1} \left(\frac{i}{n}\right)^2 \left(\frac{1}{n}\right) \\
&= \lim_{n\to\infty} \left(\frac{1}{n^3}\right) \sum_{i=0}^{n-1} i^2 \\
&= \lim_{n\to\infty} \left(\frac{1}{n^3}\right) \left(\frac{(n-1)n(2n-1)}{6}\right) \\
&= \lim_{n\to\infty} \left(\frac{2n^3 - 3n^2 + n}{6n^3}\right) \\
&= \frac{1}{3}
\end{aligned}$$

■