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Solution to Problem 1039.

Proof: Suppose $n = a + (a + 1) + (a + 2) + \cdots + (a + k)$. Then

$$\begin{aligned}n &= a + (a + 1) + (a + 2) + \cdots + (a + k) \\n &= \frac{(2a + k)(k + 1)}{2} \\2n &= k^2 + (2a + 1)k + 2a \\0 &= k^2 + (2a + 1)k + (2a - 2n)\end{aligned}$$

Solving for k (a positive integer) gives

$$k = \frac{-(2a + 1) + \sqrt{(2a + 1)^2 - 4(2a - 2n)}}{2}$$

We are interested in the number of such solutions. Obviously, the quantity inside the square root must be a perfect square. $(2a + 1)^2 - 4(2a - 2n) = (2a - 1)^2 + 8n = A^2$ for some integer A . Let's further denote $B = 2a - 1$ (i.e. B is odd). It follows that A is odd too. Then we have $8n = A^2 - B^2 = (A + B)(A - B)$. Equivalently, $2n = \left(\frac{A+B}{2}\right) \left(\frac{A-B}{2}\right)$.

Let's say $2n = \alpha\beta$ where $\alpha = \frac{A+B}{2}$ and $\beta = \frac{A-B}{2}$. Then $A = \alpha + \beta$ and $B = \alpha - \beta$. An observation from these two equalities is that α and β are of opposite parity, one is odd and the other even.

Once we reach that conclusion, we say α and β gives a solution to $2n = \alpha\beta$ if and only if one of α and β is odd and the other even. If d is an odd divisor of n , say $\alpha = d$, then $\beta = \frac{2n}{d}$. (Obviously, $\alpha > \beta$ by assumption. But by the symmetric property of $2n = \alpha\beta$. If $d < \frac{2n}{d}$, we just call $\beta = d$ and $\alpha = \frac{2n}{d}$.)

And it follows naturally that each of the odd divisors of n gives a solution to k . More specifically, if $\alpha > \beta$ are solutions, then $a = \frac{\alpha - \beta + 1}{2}$ and $k = \beta - 1$.

Therefore, the number of representations of n as the sum of consecutive integers is the same as the number of odd divisors of n . ■