Rex H. Wu
Brooklyn, NY

Solution to Problem 1040.

Let $a_n = \sum_{i=0}^{n} \frac{1}{n+2i}$, show a_n is decreasing.

$$a_n = \frac{1}{n} + \frac{1}{n+2} + \frac{1}{n+4} + \cdots + \frac{1}{3n}$$

$$a_{n+1} = \frac{1}{n+1} + \frac{1}{n+3} + \frac{1}{n+5} + \cdots + \frac{1}{3n+3}$$

We need to show $a_n - a_{n+1} > 0$.

$$\begin{aligned} a_n - a_{n+1} &= \frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \frac{1}{n+3} + \frac{1}{n+4} - \frac{1}{n+5} \\ &\quad + \cdots + \frac{1}{3n} - \frac{1}{3n+1} - \frac{1}{3n+3} \\ &= \frac{1}{n(n+1)} + \frac{1}{(n+2)(n+3)} + \frac{1}{(n+4)(n+5)} \\ &\quad + \cdots + \frac{1}{(3n)(3n+1)} - \frac{1}{3n+3} \\ &= \sum_{i=0}^{n} \frac{1}{(n+2i)(n+2i+1)} - \frac{1}{3n+3}. \end{aligned}$$

In other words, if we can show $\sum_{i=0}^{n} \frac{1}{(n+2i)(n+2i+1)} > \frac{1}{3n+3}$, we are done.

Note that

$$\begin{aligned} \sum_{i=0}^{n} \frac{1}{(n+2i)(n+2i+1)} &= \sum_{i=1}^{3n} \frac{1}{i(i+1)} - \sum_{i=1}^{n-1} \frac{1}{i(i+1)} \\ &\quad - \sum_{i=0}^{n-1} \frac{1}{(n+2i+1)(n+2i+2)}. \end{aligned}$$

Since $\dfrac{1}{(n+2i)(n+2i+1)} > \dfrac{1}{(n+2i+1)(n+2i+2)}$, for $i = 0, 1, 2, \ldots$, we know $\displaystyle\sum_{i=0}^{n} \frac{1}{(n+2i)(n+2i+1)} > \frac{1}{2}\left[\sum_{i=1}^{3n} \frac{1}{i(i+1)} - \sum_{i=1}^{n-1} \frac{1}{i(i+1)}\right]$.

We also know $\displaystyle\sum_{i=1}^{3n} \frac{1}{i(i+1)} = 1 - \frac{1}{3n+1}$ and $\displaystyle\sum_{i=1}^{n-1} \frac{1}{i(i+1)} = 1 - \frac{1}{n}$, therefore we have

$$
\begin{aligned}
\sum_{i=0}^{n} \frac{1}{(n+2i)(n+2i+1)} &> \frac{1}{2}\left[\sum_{i=1}^{3n} \frac{1}{i(i+1)} - \sum_{i=1}^{n-1} \frac{1}{i(i+1)}\right] \\
&= \frac{1}{2}\left[(1 - \frac{1}{3n+1}) - (1 - \frac{1}{n})\right] \\
&= \frac{2n+1}{2n(3n+1)}
\end{aligned}
$$

Of course, $\dfrac{2n+1}{2n(3n+1)} > \dfrac{1}{3n+3}$ by simple arithmetics. And we just showed $a_n > a_{n+1}$.

Now let's show $\displaystyle\lim_{n\to\infty} a_n > \frac{1}{2}$.

Let's pair up the sequence a_n as follow: the first term with the last term, the second with the second to the last, etc. (i.e. the denominators add up to $4n$) And we add these pairs together.

$$
\overbrace{\frac{1}{n} + \underbrace{\frac{1}{n+2} + \overbrace{\frac{1}{n+4} + \cdots + \frac{1}{3n-4}} + \frac{1}{3n-2}} + \frac{1}{3n}}
$$

Or,

$$
\begin{aligned}
a_n &= \left(\frac{1}{n} + \frac{1}{3n}\right) + \left(\frac{1}{n+2} + \frac{1}{3n-2}\right) + \left(\frac{1}{n+4} + \frac{1}{3n-4}\right) + \cdots \\
&= \left(\frac{1}{2n-n} + \frac{1}{2n+n}\right) + \left(\frac{1}{2n-(n-2)} + \frac{1}{2n+(n-2)}\right) \\
&\quad + \left(\frac{1}{2n-(n-4)} + \frac{1}{2n+(n-4)}\right) + \cdots
\end{aligned}
$$

If n is odd, then there are $(n+1)$ terms in a_n, with $\frac{1}{2n-1}$ and $\frac{1}{2n+1}$ as the inner most pair. Their sum is $\frac{4n}{4n^2-1}$ which is greater than $\frac{1}{n}$.

Also note that for any $\alpha > 1$, $\frac{1}{2n-\alpha} + \frac{1}{2n+\alpha} = \frac{4n}{4n^2-\alpha^2} > \frac{4n}{4n^2-1} > \frac{1}{n}$.

To finish up, there are $\frac{n+1}{2}$ pairs, and each pair sums up to greater than $\frac{1}{n}$. Therefore,

$$a_n = \sum_{i=0}^{n} \frac{1}{n+2i} > \frac{n+1}{2} \cdot \frac{1}{n} > \frac{1}{2}.$$

The argument is exactly the same if n is even. In that case, there are $(n{+}1)$ term, with $\frac{1}{2n}$ as the middle term. We can then rearrange the sequence as

$$\begin{aligned} a_n &= \left(\frac{1}{n} + \frac{1}{3n}\right) + \left(\frac{1}{n+2} + \frac{1}{3n-2}\right) + \left(\frac{1}{n-4} + \frac{1}{3n-4}\right) \\ &\quad + \cdots + \left(\frac{1}{2n-2} + \frac{1}{2n+2}\right) + \left(\frac{1}{2n}\right) \end{aligned}$$

Again, notice that for any $\alpha > 2$, $\frac{1}{2n-\alpha} + \frac{1}{2n+\alpha} = \frac{4n}{4n^2-\alpha^2} > \frac{1}{2n-2} + \frac{1}{2n+2} = \frac{4n}{4n^2-4} > \frac{1}{n}$.

Since there are $\frac{n}{2}$ pairs, we have

$$a_n > \frac{n}{2} \cdot \frac{1}{n} + \frac{1}{2n} > \frac{1}{2}.$$

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