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Solution to Problem 1041.

Let's assume the quarter circle has a radius of 2 units.

Then the radius $r$ of each of the small circles (the ones in a group of four) is $\frac{1}{4}$. The radius $R$ of the each of the big circles inscribed in the right isosceles triangles is $R = \frac{K}{s}$, where $K = \frac{1}{2}$ is the area of the triangle, $s = \frac{2+\sqrt{2}}{2}$ is the semi-perimeter of the triangle. Solving for $R$ gives $R = \frac{2-\sqrt{2}}{2}$.

The large circle tangent to the arc of the quarter circle is obviously $R = \frac{2-\sqrt{2}}{2}$.

To look for the radius of the small circles tangent to the quarter circle, look at the accompanying figure. The following relationships can be found.

$$\begin{aligned}
r + c + d &= 2 && (1)\\
\frac{r}{\sqrt{2}} &= \frac{a}{b} && (2)\\
\frac{r}{\sqrt{2}} &= \frac{c}{d} && (3)\\
c^2 &= a^2 + r^2 && (4)\\
d^2 &= b^2 + 2 && (5)\\
(a+b)^2 &= \left(\frac{2-\sqrt{2}}{2} + r\right)^2 - \left(\frac{2-\sqrt{2}}{2} - r\right)^2 &&\\
&= (4 - 2\sqrt{2})r && (6)
\end{aligned}$$

From (2), $b = \frac{\sqrt{2}a}{r}$. Substitute that into (6) to get $a^2 = \frac{(4-2\sqrt{2})r^3}{r^2+2\sqrt{2}r+2}$.

From (3), we know $d = \frac{\sqrt{2}c}{r}$. Substitute that into (1) to get $c = \frac{2r-r^2}{r+\sqrt{2}}$ or $c^2 = \frac{r^4-4r^3+4r^2}{r^2+2\sqrt{2}r+2}$.

Finally, substitute $a^2$ and $c^2$ into (4) to get

$$\frac{r^4 - 4r^3 + 4r^2}{r^2 + 2\sqrt{2}r + 2} = \frac{(4 - 2\sqrt{2})r^3}{r^2 + 2\sqrt{2}r + 2} + r^2$$

$$
\begin{aligned}
r^4 - 4r^3 + 4r^2 &= (4 - 2\sqrt{2})r^3 + r^4 + 2\sqrt{2}r^3 + 2r^2 \\
2r^2 &= 8r^3 \\
r &= \frac{1}{4}
\end{aligned}
$$

And this completes the proof.

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