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Solution to Problem 1043.

Suppose  $\gcd(x, v) = A$ ,  $\gcd(y, u) = D$ ,  $x = AB$ ,  $y = DE$ ,  $u = DF$  and  $v = AC$  for some integers  $A, B, C, D, E$  and  $F$ .

Then  $\gcd(x, y) = 1 \Rightarrow \gcd(A, D) = \gcd(A, E) = \gcd(B, D) = \gcd(B, E) = 1$ .  $\gcd(x, v) = A \Rightarrow \gcd(B, C) = 1$ .  $\gcd(u, v) = 1 \Rightarrow \gcd(C, D) = \gcd(C, F) = \gcd(A, F) = 1$ . And finally  $\gcd(y, u) = D \Rightarrow \gcd(E, F) = 1$ .

From  $xy = uv$ , we have  $(AB)(DE) = (AC)(DF)$ , or  $BE = CF$ . Since  $\gcd(B, C) = 1$  and  $\gcd(E, F) = 1$ , we can only have  $B = F$  and  $C = E$ .

$$\begin{aligned}x - y &= u + v \\x - v &= y + u \\A(B - C) &= D(E + F) \\A(B - C) &= D(C + B)\end{aligned}$$

Since  $\gcd(A, D) = 1$ , we can only have  $A|(B + C)$  and  $D|(B - C)$ . In other words, if  $\alpha A = (B + C)$ , then  $(B - C) = \alpha D$ .

By adding and subtracting the two equations, we get  $\alpha(A + D) = 2B$  and  $\alpha(A - D) = 2C$ . Then  $\alpha = 1$  or  $\alpha = 2$ . If  $\alpha \geq 3$ , then  $\gcd(B, C) \neq 1$ , a contradiction.

**Case (i)**  $\alpha = 1$

We have  $A + D = 2B$  and  $A - D = 2C$ . Or  $A = D + 2C$ . Here  $C$  can be any positive integer,  $C = 1, 2, 3, 4, \dots$ . Since  $\gcd(A, D) = 1$ ,  $D$  cannot be even. We also know  $\gcd(D, C) = 1$ . Therefore,  $D$  can be any integer with the restriction that  $\gcd(D, 2C) = 1$ .  $A = D + 2C$  and  $B = (A + D)/2 = C + D$ .

Since  $x = AB = (D + 2C)(C + D) = D^2 + 2C^2 + 3CD$ ,  $y = CD$ ,  $u = \max\{AC, BD\} = \max\{(D + 2C)C, (C + D)D\} = \max\{CD + 2C^2, CD + D^2\}$  and  $v = \min\{AC, BD\} = \min\{CD + 2C^2, CD + D^2\}$  (The *max* and *min* functions are used to satisfy the condition  $u > v$ .), we have our first class of solutions:  $(x, y, u, v) = (D^2 + 2C^2 + 3CD, CD, \max\{CD + 2C^2, CD + D^2\}, \min\{CD + 2C^2, CD + D^2\})$ , with  $C = 1, 2, 3, 4, \dots$  and  $\gcd(D, 2C) = 1$ .

**Case (ii)  $\alpha = 2$**

We have  $2(A + D) = 2B$  and  $2(A - D) = 2C$ . Or  $A = D + C$ . Note that  $A$  and  $D$  are of the opposite parity. Otherwise,  $\gcd(B, C) \neq 1$ . This implies  $C$  must be odd,  $C = 2i - 1$  for  $i = 1, 2, 3, 4, \dots$ .  $D$  can be any integer with  $\gcd(C, D) = 1$ .  $A = D + C$  and  $B = A + D = C + 2D$ .

Again, with  $x = AB = C^2 + D^2 + 3CD$ ,  $y = CD$ ,  $u = \max\{AC, BD\} = \max\{CD + C^2, CD + 2D^2\}$  and  $v = \min\{AC, BD\} = \min\{CD + C^2, CD + 2D^2\}$ , our second class of solutions therefore is  $(x, y, u, v) = (C^2 + D^2 + 3CD, CD, \max\{CD + C^2, CD + 2D^2\}, \min\{CD + C^2, CD + 2D^2\})$ , with  $C = 1, 3, 5, 7, \dots$  and  $\gcd(D, C) = 1$ .

But this is the exact expression as in (i) with  $C$  and  $D$  switched.

In conclusion, the solution is:

$(x, y, u, v) = (D^2 + 2C^2 + 3CD, CD, \max\{CD + 2C^2, CD + D^2\}, \min\{CD + 2C^2, CD + D^2\})$ , with  $\gcd(D, 2C) = 1$ .

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