Rex H. Wu
Brooklyn, NY
RexHWu@aol.com

Solution to Problem 1044.

Euler's constant is defined as

$$\gamma = \lim_{n\to\infty}\left(\sum_{i=1}^{n}\frac{1}{i} - \ln n\right) = \lim_{n\to\infty}\left(H_n - \ln n\right).$$

Let's rewrite the Harmonic series as,

$$\lim_{n\to\infty} H_n = \gamma + \lim_{n\to\infty}\ln n.$$

Then,

$$\begin{aligned}
\lim_{n\to\infty}\frac{n}{\ln n}\sum_{i=1}^{n-1}\frac{1}{ni-i^2} &= \lim_{n\to\infty}\frac{n}{\ln n}\sum_{i=1}^{n-1}\frac{1}{i(n-i)} \\
&= \lim_{n\to\infty}\frac{n}{\ln n}\left[\frac{1}{n}\sum_{i=1}^{n-1}\left(\frac{1}{i}+\frac{1}{n-i}\right)\right] \\
&= \lim_{n\to\infty}\frac{2H_{n-1}}{\ln n} \\
&= \lim_{n\to\infty}\frac{2(H_n-1/n)}{\ln n} \\
&= \lim_{n\to\infty}\frac{2H_n}{\ln n} - \lim_{n\to\infty}\frac{2}{n\ln n} \\
&= \lim_{n\to\infty}\frac{2(\gamma+\ln n)}{\ln n} \\
&= \lim_{n\to\infty}\frac{2\gamma}{\ln n} + \lim_{n\to\infty}\frac{2\ln n}{\ln n} \\
&= 2
\end{aligned}$$

■