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Solution to Problem 1045.

Proof: Let's review some of the basics. Suppose G is a finite group and H is a subgroup of G , then $|H|$ divides $|G|$. Furthermore, if G is abelian and q divides $|G|$, then there is a subgroup of order q .

Let's say $G = \{e, a_2, a_3, \dots, a_{2n}\}$ with identity e is our group of interest, where n is an odd number. We know there is a member, say a_2 , such that $a_2^2 = e$. There is also another member whose order is m where m is odd and $m \mid n$.

Suppose a_2 is not the only member with an order of 2. Say a_3 , such that $a_3^2 = e$. Let's call this set $H_1 = \{e, a_2, a_3\}$. By the commutative property, we have $(a_2a_3)(a_2a_3) = a_2^2a_3^2 = e$. Clearly, $a_2a_3 \neq a_2$ since this implies $a_3 = e$. By the same reasoning, $a_2a_3 \neq a_3$ and $a_2a_3 \neq e$. Therefore, $a_2a_3 \notin H_1$.

Let's say $a_2a_3 = a_4$ and $H_2 = \{e, a_2, a_3, a_4\}$. H_2 may form the Klein 4-group. But H_2 cannot be a subgroup of G because $4 \nmid 2n$. Therefore, $a_2a_4 = a_5 \notin H_2$ and $a_3a_4 = a_6 \notin H_2$. Also note that $a_5^2 = a_6^2 = e$.

The question is can the set $H_{2m} = \{e, a_2, a_3, \dots, a_{2m}\}$ with m odd and $m \mid n$ form a subgroup? Obviously, by the same reasoning as above, we have $a_2^2 = a_3^2 = \dots = a_{2m}^2 = e$. Since we know there is also a member in H_{2m} whose order is m , we have a contradiction. Therefore, H_{2m} cannot form a subgroup.

By the same token, $G = H_{2n}$ cannot form a group because every member of G has order 2 and yet there is another member whose order is m . Thus, there is only one subgroup whose order is 2. ■