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Solution to Problem 1051.

Part(1)

Suppose the square ABCD has sides with unit length. Connect the midpoints $A'B'$, $B'C'$, $C'D'$ and $D'A'$ as shown.

To see that the lines AB' and AC'' trisect the line $D'A'$, extend line AB to B'' so that $BB'' = AB/2$ and extend line AD to D'' so that $DD'' = AD/2$. Connect points B'' and D''. Then $\triangle AA'D' \sim \triangle AB''D''$ with a similitude of 1:3.

After a $\pi/4$ rotation of square $A'B'C'D'$, it is obvious that square $A'B'C'D'$ has the same construction as square $EFGH$. Therefore the octagons are similar.

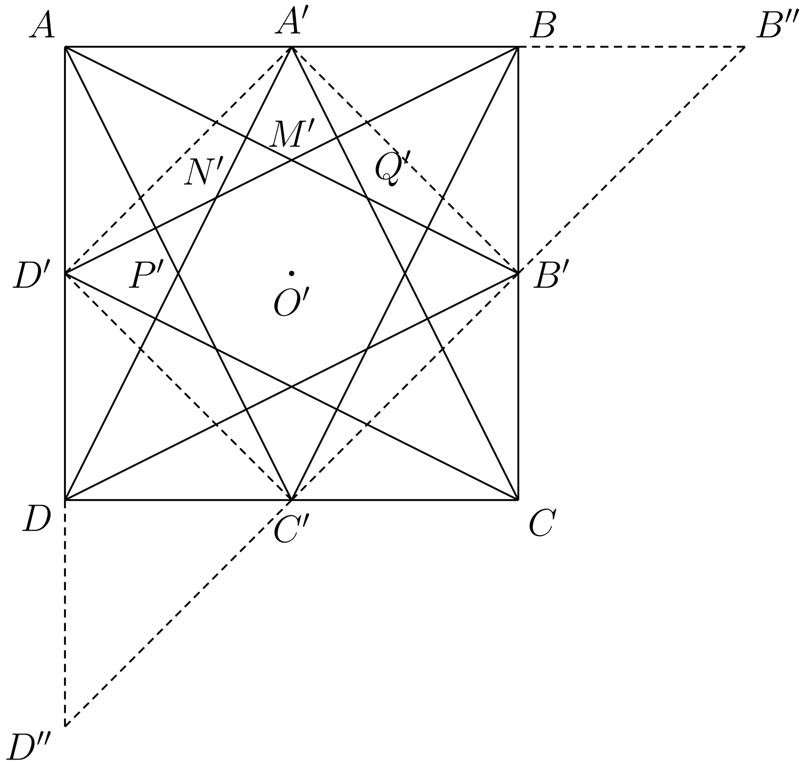
Observe that $O'M' = O'P' = O'A'/2 = 1/4$ and $O'N' = O'Q' = O'A/3 = \sqrt{2}/6$ where O' is the center of the octagon. ($OM = OP = OE/2 = \sqrt{2}/4$ and $ON = OQ = 1/3$ in the larger octagon. Again, O is the center.) Since $\angle P'O'N' = \angle N'O'M' = \angle M'O'Q' = \pi/4$, it follows that $P'N' = N'M' = M'Q'$. Therefore, the octagon is equilateral. Since $\triangle M'O'N'$ is not isosceles, the octagon is not equiangular.

Alternatively, notice that there are two essentially different types of vertices on the octagon. Namely, M' and N'. P', P and M are the same as M'; and N, Q' and Q are the same as N'. It is easy to see that $\angle AM'B \ncong \angle EQF$. Thus, vertices M' and Q (or the equivalent of vertex Q, N') are different. So the octagon is not equiangular.

By the same token, there are two different pairs of sides in the octagon, namely, $P'N' \cong N'M'$, and $N'M' \cong M'Q'$. But then it follows that $P'N' \cong M'Q'$. And the octagon is equilateral.

Part (2)

Since square $A'B'C'D'$ is exactly half the size of square $ABCD$ or square $EFGH$, the ratio of the octagons is 1:2.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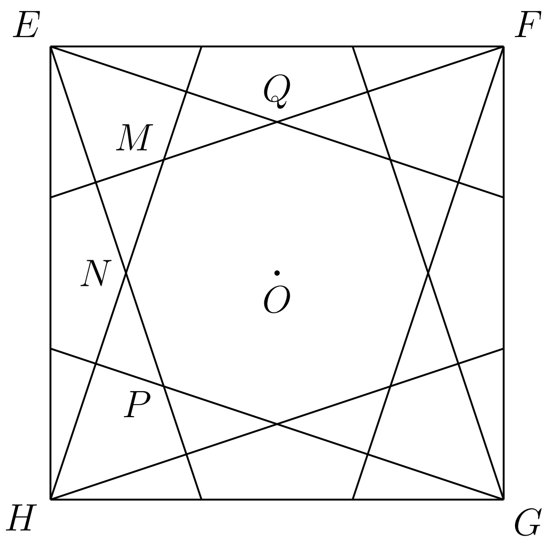
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