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Solution to Problem 1053.

We start with the gamma function and use the substitution $t = ru$,

$$\begin{aligned}\Gamma(x) &= \int_0^\infty t^{x-1}e^{-t}dt \\ &= \int_0^\infty (ru)^{x-1}e^{-ru}r du \\ &= r^x \int_0^\infty u^{x-1}e^{-ru}du\end{aligned}$$

Equivalently,

$$\Gamma(x)\frac{1}{r^x} = \int_0^\infty u^{x-1}e^{-ru}du$$

Multiply the last expression by $(-1)^{r-1}$ on both sides,

$$\Gamma(x)\frac{(-1)^{r-1}}{r^x} = (-1)^{r-1}\int_0^\infty u^{x-1}e^{-ru}du$$

Summing the above expression gives

$$\begin{aligned}\Gamma(x)\sum_{r=1}^\infty \frac{(-1)^{r-1}}{r^x} &= \sum_{r=1}^\infty (-1)^{r-1}\int_0^\infty u^{x-1}e^{-ru}du \\ &= \int_0^\infty u^{x-1}\sum_{r=1}^\infty (-1)^{r-1}e^{-ru}du\end{aligned}$$

Since $\sum_{r=1}^\infty \frac{(-1)^{r-1}}{r^x} = (1-2^{1-x})\sum_{r=1}^\infty \frac{1}{r^x} = (1-2^{1-x})\zeta(x)$ and $\sum_{r=1}^\infty (-1)^{r-1}e^{-ru}$ $= \frac{1}{1+e^u}$, we have

$$\int_0^\infty \frac{u^{x-1}}{1+e^u}du = (1-2^{1-x})\Gamma(x)\zeta(x)$$

Our problem asks for two special cases of the above integral, namely, $x = 2$ and $x = 4$. For $x = 2$,

$$\begin{aligned}\int_0^\infty \frac{u}{1+e^u}du &= (1-2^{1-2})\Gamma(2)\zeta(2) \\ &= \left(\frac{1}{2}\right)(1!)\left(\frac{\pi^2}{6}\right) \\ &= \frac{\pi^2}{12}.\end{aligned}$$

For $x = 4$,

$$\begin{aligned}\int_0^\infty \frac{u^3}{1+e^u}du &= (1-2^{1-4})\Gamma(4)\zeta(4) \\ &= \left(\frac{7}{8}\right)(3!)\left(\frac{\pi^4}{90}\right) \\ &= \frac{7\pi^4}{120}.\end{aligned}$$

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Gradshteyn and Ryzhik give two general forms of the above

$$\int_0^\infty \frac{x^{2n-1}}{e^{px}+1}dx = \left(1-2^{1-2n}\right)\left(\frac{2\pi}{p}\right)^{2n}\frac{|B_{2n}|}{4n}$$

where $B_{2n}$ is the Bernoulli number. And

$$\int_0^\infty \frac{x^{n-1}e^{-px}}{1+e^x}dx = (n-1)!\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(p+k)^n}.$$

References:


1. Havil, J. *Gamma Exploring Euler's Constant*, pp. 59-60; Princeton University Press, 2003.

2. Gradshteyn I.S. and Ryzhik I.M., *Table of Integrals, Series, and Products* 6th Ed., p. 349; Academic Press, 2000.