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Solution to Problem 1061.

The result follows naturally once we recognize that

$$x = \sum_{k=0}^{n} 2^k \binom{2n+1}{2k+1}$$

and

$$y = \sum_{k=0}^{n} 2^k \binom{2n+1}{2k}$$

are solutions to the equation $y^2 - 2x^2 = -1$, which is closely related to Pell's equation.

Rearranging $y^2 - 2x^2 = -1$, we have

$$\frac{x^2 - 1}{2} = \left(\frac{y-1}{2}\right)\left(\frac{y+1}{2}\right).$$

Since $y = \sum_{k=0}^{n} 2^k \binom{2n+1}{2k} = 1 + \sum_{k=1}^{n} 2^k \binom{2n+1}{2k} = 1 + 2 \cdot \sum_{k=1}^{n} 2^{k-1} \binom{2n+1}{2k}$, which is an odd number. $(\frac{y-1}{2})$ and $(\frac{y+1}{2})$ are two consecutive integers.

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