1077. *Proposed by Robert C. Gebhardt, Hopatcong, NJ and the Editors.* Robert Gebhhardt made the following proposal:

Consider a prime number p. Add its digits to get another number. If this sum has more than one digit, add them to get another number, and so on until you get a single digit (call it $R(p)$, the digital root.) For example, if $p = 199$ then $1+9+9 = 19$, $1+9 = 10$, and $1+0 = 1$, so $R(199) = 1$.

Obviously, $R(p)$ cannot equal 0. Is there a prime number p for which $R(p) =$ 6 or 9? (Note: All other possible digital sums do occur.)

We have generalized the problem:

Consider a whole number $n > 2$. If x is any number, let $R_n(x)$ be the digital root of x in base n. (All numerical representations are in base n.) An example from base five: if $x = 1123_{\text{five}}$ then $1+1+2+3 = 12_{\text{five}}$, and $1+2 = 3_{\text{five}}$, so $R_5(1123_{\text{five}}) = 3$.

Suppose p is a prime number. Is it possible for $\gcd(R_n(p), n-1)$ to be larger than 1?

Solution by **Rex H. Wu**, *Brooklyn, NY.*

Suppose x is a positive integer in base 10 and $(a_m a_{m-1} \ldots a_2 a_1 a_0)_n$ is its representation in base n, then $x = a_m n^m + a_{m-1} n^{m-1} + \cdots + a_2 n^2 + a_1 n + a_0$. Since $n \equiv 1 \pmod{n-1}$, $x = a_m n^m + a_{m-1} n^{m-1} + \ldots a_2 n^2 + a_1 n + a_0 \equiv a_m + a_{m-1} + \ldots a_2 + a_1 + a_0 = R_n(x) \equiv N \pmod{n-1}$ with $N \in \{0, 1, 2, \ldots, n-2\}$. Because the digital root $R_n(x)$ cannot be 0, we can rewrite the above expression as

$$R_n(x) = \begin{cases} N & \text{if } N \not\equiv 0 \pmod{n-1} \\ n-1 & \text{if } N \equiv 0 \pmod{n-1} \end{cases}$$

For the Editors' question, can $\gcd(R_n(p), n-1)$ be larger than 1?

The answer is a trivial yes. Suppose the base is n with $n > 2$, and $(n-1) = p_1^{e_1} p_2^{e_2} \cdots p_m^{e_m}$ for some primes $p_1, p_2, \ldots, p_m$ and some positive integers e_i with $i \in \{1, 2, 3, \ldots, m\}$. Then $\gcd(R_n(p_i), n-1) = \gcd(p_i, n-1) = p_i$, for some prime $p_i \in \{p_1, p_2, p_3, \ldots, p_m\}$.

However, if we limit the prime p to be greater than the base n, then the answer is no. Let's say there is a prime $p > n$ such that $\gcd(R_n(p), n-1) = m > 1$. Then $R_n(p) = km$ for some $k \in \mathbb{N}$, $km < (n-1)$ and $\gcd(k, n-1)$=1. It follows that $p = t(n-1) + km$ for some $t \in \mathbb{N}$. But $\gcd(R_n(p), n-1) = m$ implies m divides $(n-1)$. Then $p = mM$ for some integer M, which contradicts the assumption that p is prime.

Now, in Gebhardt's question, we just have to let $n = 10$, $m = 3$ (or 9). Then $k = 2$ (or 1 respectively). The answer is an obvious no since $R(p) = 6$ implies 3 divides p (or $R(p) = 9$ implies 9 divides p).

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