**1078.** *Proposed by Robert C. Gebhardt, Hopatcong, NJ.*
Let $a$ and $b$ be positive real numbers. Consider the Fibonacci-type sequence $S_1, S_2, \ldots, S_n, \ldots$ where $S_{n+2} = aS_n + bS_{n+1}$ for $n \geq 1$. For any real numbers $S_1$ and $S_2$, both not zero, find

$$\lim_{n\to\infty} \frac{S_n}{S_{n+k}}$$

for $k = 1, 2, 3, \ldots$

*Solution by* **Rex H. Wu**, *Brooklyn, NY.*
The sequence $S_{n+2} = aS_n + bS_{n+1}$ can be represented by the difference equation

$$\lambda^{n+2} = a\lambda^n + b\lambda^{n+1}$$

For $\lambda \neq 0$, solving for $\lambda$ to get

$$\lambda^2 - b\lambda - a = 0$$

$$\lambda_1 = \frac{b + \sqrt{b^2 + 4a}}{2}$$
$$\lambda_2 = \frac{b - \sqrt{b^2 + 4a}}{2}$$

$S_n$ can be expressed as the linear combination of $\lambda_1$ and $\lambda_2$.

$$S_n = \alpha\lambda_1^n + \beta\lambda_2^n$$

From the above, let's calculate the limit of $S_n/S_{n+k}$.

$$\begin{aligned}\lim_{n\to\infty} \frac{S_n}{S_{n+k}} &= \lim_{n\to\infty} \frac{\alpha\lambda_1^n + \beta\lambda_2^n}{\alpha\lambda_1^{n+k} + \beta\lambda_2^{n+k}} \\ &= \lim_{n\to\infty} \frac{\alpha + \beta\left(\frac{\lambda_2}{\lambda_1}\right)^n}{\alpha\lambda_1^k + \beta\lambda_2^k\left(\frac{\lambda_2}{\lambda_1}\right)^n}\end{aligned}$$

Since $\lim(\lambda_2/\lambda_1)^n = 0$, we have

$$\lim_{n\to\infty} \frac{S_n}{S_{n+k}} = \frac{1}{\lambda_1^k} = \left(\frac{2}{b + \sqrt{b^2 + 4a}}\right)^k$$

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