1078. *Proposed by Robert C. Gebhardt, Hopatcong, NJ.*
Let a and b be positive real numbers. Consider the Fibonacci-type sequence $S_1, S_2, \ldots, S_n, \ldots$ where $S_{n+2} = aS_n + bS_{n+1}$ for $n \geq 1$. For any real numbers S_1 and S_2, both not zero, find

$$\lim_{n\to\infty} \frac{S_n}{S_{n+k}}$$

for $k = 1, 2, 3, \ldots$

Solution by **Rex H. Wu**, *Brooklyn, NY.*
The sequence $S_{n+2} = aS_n + bS_{n+1}$ can be represented by the difference equation

$$\lambda^{n+2} = a\lambda^n + b\lambda^{n+1}$$

For $\lambda \neq 0$, solving for λ to get

$$\lambda^2 - b\lambda - a = 0$$

$$\lambda_1 = \frac{b + \sqrt{b^2 + 4a}}{2}$$
$$\lambda_2 = \frac{b - \sqrt{b^2 + 4a}}{2}$$

S_n can be represented as the linear combination of λ_1 and λ_2.

$$S_n = \alpha\lambda_1^n + \beta\lambda_2^n$$

Along with the initial conditions S_0 and S_1 gives $S_0 = \alpha + \beta$ and $S_1 = \alpha\lambda_1 + \beta\lambda_2$. Solving for α and β gives $\alpha = (S_1 - \lambda_2 S_0)/(\lambda_1 - \lambda_2)$ and $\beta = (\lambda_1 S_0 - S_1)/(\lambda_1 - \lambda_2)$. However, this is irrelevant to our solution.

From the above, let's calculate the limit of S_n/S_{n+1}.

$$\begin{aligned}
\lim_{n\to\infty} \frac{S_n}{S_{n+1}} &= \lim_{n\to\infty} \frac{\alpha\lambda_1^n + \beta\lambda_2^n}{\alpha\lambda_1^{n+1} + \beta\lambda_2^{n+1}} \\
&= \lim_{n\to\infty} \frac{\alpha + \beta\left(\frac{\lambda_2}{\lambda_1}\right)^n}{\alpha\lambda_1 + \beta\lambda_2\left(\frac{\lambda_2}{\lambda_1}\right)^n}
\end{aligned}$$

Since $\lim(\lambda_2/\lambda_1)^n = 0$, we have

$$\lim_{n\to\infty} \frac{S_n}{S_{n+1}} = \frac{1}{\lambda_1} = \frac{2}{b + \sqrt{b^2 + 4a}}$$

To find $\lim S_n/S_{n+k}$, notice that

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\begin{aligned}
\lim_{n\to\infty} \frac{S_n}{S_{n+k}} &= \lim_{n\to\infty} \left(\frac{S_n}{S_{n+1}}\right)\left(\frac{S_{n+1}}{S_{n+2}}\right)\cdots\left(\frac{S_{n+k-1}}{S_{n+k}}\right) \\
&= \prod_{i=0}^{k-1} \lim_{n\to\infty} \left(\frac{S_{n+i}}{S_{n+i+1}}\right) \\
&= \left(\frac{2}{b+\sqrt{b^2+4a}}\right)^k \\
&= \frac{1}{\lambda_1^k}
\end{aligned}
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