1080. *Proposed by Brian Smith, Utica, MN.*

A deck of nine cards can be numbered so that the sum of the numbers on a randomly chosen pair of cards totals to an integer from 2 to 12 with the same frequency as rolling two standard dice. What the the numbers on the nine cards? Is the solution unique?

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Let $\mathbb{S} = \{2, 3, 3, 4, 4, 4, 5, 5, 5, 5, 6, 6, 6, 6, 6, 7, 7, 7, 7, 7, 7, 8, 8, 8, 8, 8, 9, 9, 9, 9, 10, 10, 10, 11, 11, 12\}$ be the set of numbers generated by two dice. Let $\mathbb{A} = \{a_1, a_2, a_3,\ a_4, a_5, a_6, a_7, a_8, a_9\}$ be the nine numbers that have the property as stated in the problem. Let's also assume that the a_i's in $\mathbb{A}$ is in a non-decreasing order. Obviously, there are 36 ways to choose two numbers from $\mathbb{A}$, namely, a_i and a_j with $i \neq j$. Each pair $a_i + a_j$ sums up to a number correspond to exactly one number in $\mathbb{S}$. If we add all the $a_i + a_j$ with $i \neq j$ and all the numbers in $\mathbb{S}$, we have

$$8 \sum_{k=1}^{9} a_k = 252$$

Or simply, $a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 = 31.5$. From this last equality, we can conclude that a_i is of the form $a_i = n_i + 0.5$ where $n_i \in \mathbb{Z}$.

Since $\sum a_i = 31.5$, there is a number, $a_k = n_k + \varepsilon_k$ where $n_k \in \mathbb{Z}$ and $0 < \varepsilon_k < 1$. Then it is obvious that $a_i \notin \mathbb{Z}$ since if there is a a_i as an integer, then $a_i + a_k \notin \mathbb{S}$. That means $a_i = n_i + \varepsilon_i$ for some $0 < \varepsilon_i < 1$. Furthermore, for any $i, j = 1, 2, 3, \ldots, 9$, $\varepsilon_i + \varepsilon_j = 1$. The only number that satisfies this condition is $\varepsilon_i = 0.5$ for $i = 1, 2, 3, \ldots, 9$. That is $a_i = n_i + 0.5$, for some $n_i \in \mathbb{Z}$. Using similar arguments, there is no complex numbers that would give the set $\mathbb{S}$ (in this case, the imaginary parts add up to 0).

Next, look at $a_1 + a_2 = 2$. One obvious solution is $a_1 = 0.5$ and $a_2 = 1.5$. Since there is only one 2 in $\mathbb{S}$, $a_1 = 0.5$ and $a_2 = 1.5$ are unique in $\mathbb{A}$.

For the number 3, $3 = 0.5 + 2.5$ or $3 = 1.5 + 1.5$. Since $a_2 = 1.5$ is unique, we must have $a_3 = a_4 = 2.5$.

Similarly, for the number 4, $4 = 0.5 + 3.5$ or $4 = 1.5 + 2.5$. There are three 4's in $\mathbb{S}$, we already have $a_2 + a_3 = 4$ and $a_2 + a_4 = 4$, we need a $a_5 = 3.5$ which gives $a_1 + a_5 = 4$.

For the number 5, we have $5 = 0.5 + 4.5$, $5 = 1.5 + 3.5$ and $5 = 2.5 + 2.5$. There are four 5's. We already have $a_2 + a_5 = 5$, $a_3 + a_4 = 5$. We can have two possibilities for a_6 and a_7, either $a_6 = 3.5$ and $a_7 = 4.5$ or $a_6 = a_7 = 4.5$. $a_6 = 3.5$ leads to an extra $a_2 + a_6 = 5$. Hence, $a_6 = a_7 = 4.5$.

The number 6 is also similar. $6 = 0.5 + 5.5$, $6 = 1.5 + 4.5$ and $6 = 2.5 + 3.5$. We know $a_2 + a_6 = a_2 + a_7 = 6$ and $a_3 + a_5 = a_4 + a_5 = 6$. We need one more way of generating 6. $a_8 = 5.5$ would do, giving $a_1 + a_8 = 6$, exactly five ways representing 6. Again, for the same reason as in the case for the number 5, we cannot have $a_8 = 2.5$ or 4.5, otherwise, we would have extra 3's, 4's or 5's, etc.

For 7 is the same. $7 = 0.5 + 6.5$, $7 = 1.5 + 5.5$, $7 = 2.5 + 4.5$ and $7 = 3.5 + 3.5$. We already have five ways representing 7, namely, $a_2 + a_8 = a_3 + a_6 = a_3 + a_7 = a_4 + a_6 = a_4 + a_7 = 7$. Adding $a_9 = 6.5$ completes the the set

$\mathbb{A}$. $\mathbb{A} = \{0.5, 1.5, 2.5, 2.5, 3.5, 4.5, 4.5, 5.5, 6.5\}$. It is clear that the sum of the members in $\mathbb{A}$ is 31.5.

Now it is a matter of checking to see if the set would satisfy the remaining numbers, 8, 9, 10, 11 and 12.

For 8, $8 = a_2 + a_9 = a_3 + a_8 = a_4 + a_8 = a_5 + a_6 = a_5 + a_7$, exactly 5 ways.

For 9, it checks too. $9 = a_3 + a_9 = a_4 + a_9 = a_5 + a_8 = a_6 + a_7$.

For the number 10, $10 = a_6 + a_9 = a_6 + a_8 = a_7 + a_8$. $11 = a_6 + a_8 = a_7 + a_8$ and $12 = a_8 + a_9$.

We now have a set that satisfies the problem. Is this set unique? From the above algorithm, the uniqueness depends on the initial solution of $a_1 + a_2 = 2$. Suppose $a_1 = -n_i + 0.5$ where $n_i = 1, 2, 3, \ldots$, then $a_2 = (n_i + 1) + 0.5$. The increment of a_1 and a_2 is $(a_2 - a_1) = 2n_i + 1 \geq 3$. However, the increment between each successive pair in $\mathbb{S}$ is at most 1. This means for the numbers $3, 4$ and 5 (or above), the only way to obtain them is by adding $a_1 = -n_1 + 0.5$ to $a_3 = a_4 = n_1 + 2 + 0.5$ to obtain 3; $a_1 = -n_1 + 0.5$ to $a_5 = a_6 = a_7 = -n_1 + 3 + 0.5$; and $a_1 = -n_1 + 0.5$ to $a_8 = a_9 = -n_1 + 3 + 0.5$ to obtain 5. We already exhaust the nine numbers but meanwhile we still need two more numbers to make four 5's. Notice that we cannot use any number $a_3 = n_3 + 0.5$ such that $-n_i + 0.5 < a_3 < n_i + 0.5$, because then $a_1 + a_3 < 2$.

Therefore, the set $\mathbb{A} = \{0.5, 1.5, 2.5, 2.5, 3.5, 4.5, 4.5, 5.5, 6.5\}$ is a solution and it is unique.

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