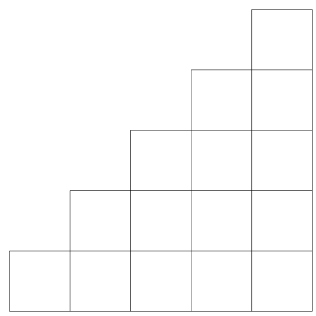
1082. *Proposed by James Rudzinski, Emory University, Atlanta, GA.*

An n-staircase is a grid of $1+2+\cdots+n=\binom{n+1}{2}$ squares arranged so that column 1 has 1 square, column 2 has 2 squares, $\ldots$, and column n has n squares. How many *rectangular* regions are bounded by the gridlines of an n staircase? The figure shows a 5-staircase.



Solution by **Rex H. Wu**, *Brooklyn, NY.*

We will consider the $i\times j$ rectangles, with $i<j$. By symmetry (namely, the $j\times i$ rectangles), suppose there are $S(n)$ such rectangles, then there are $2S(n)$ contained in the staircase.

For the $\binom{n+1}{2}$ staircase, if n is even, the rectangles that are contained in the staircase are $\{1\times 2,\ 1\times 3,\ \ldots,\ 1\times n\ \}$, $\{2\times 3,\ 2\times 4,\ \ldots,\ 2\times(n-1)\}$, $\{3\times 4,\ 3\times 5,\ \ldots,\ 3\times(n-2)\}$, $\ldots$, $\{\frac{n}{2}\times(\frac{n}{2}+1)\}$. If n is odd, then the last rectangles would be $\{\frac{n-1}{2}\times(\frac{n-1}{2}+1),\ \frac{n-1}{2}\times(\frac{n-1}{2}+2)\}$ instead of $\{\frac{n}{2}\times(\frac{n}{2}+1)\}$.

Let's arrange the above rectangles in the following manner.

For n even,

$$\begin{array}{llllll}
\frac{n}{2}\times(\frac{n}{2}+1) & & & & & \\
(\frac{n}{2}-1)\times(\frac{n}{2}+2) & (\frac{n}{2}-1)\times(\frac{n}{2}+1) & (\frac{n}{2}-1)\times\frac{n}{2} & & & \\
 & \vdots & & & & \\
2\times(n-1) & 2\times(n-2) & \ldots & 2\times 3 & & \\
1\times n & 1\times(n-1) & \ldots & 1\times 4 & 1\times 3 & 1\times 2
\end{array}$$

For n odd,

$$\begin{array}{llllll}
\frac{n-1}{2}\times(\frac{n-1}{2}+2) & \frac{n-1}{2}\times(\frac{n-1}{2}+1) & & & & \\
(\frac{n-1}{2}-1)\times(\frac{n-1}{2}+3) & \ldots & (\frac{n-1}{2}-1)\times(\frac{n-1}{2}) & & & \\
 & \vdots & & & & \\
2\times(n-1) & 2\times(n-2) & \ldots & 2\times 3 & & \\
1\times n & 1\times(n-1) & \ldots & 1\times 4 & 1\times 3 & 1\times 2
\end{array}$$

An observation from the staircase is that the number of any particular $i\times j$ rectangle form a triangular number. Let's take the 2×3 rectangle in the 6-

staircase. Starting with the third and fourth layers, namely the 1×3 and 2×3 layers, there is one 2×3 rectangle. There are two in the fourth and fifth layers and three in the fifth and sixth layers. Therefore the number of 2×3 rectangles in a 6-staircase is $1 + 2 + 3 = 6$, a triangular number.

Take for example, the arrangement for the 8-staircase is

1						
1	3	6				
1	3	6	10	15		
1	3	6	10	15	21	28

and for the 9-staircase is

1	3						
1	3	6	10				
1	3	6	10	15	21		
1	3	6	10	15	21	28	36

A recurrence relation can be obtained as follow:

For n even, the recurrence relation $S(0) = 0$, $S(2(i+1)) = S(2i) + \sum_{j=1}^{2i+1} \frac{j(j+1)}{2}$ for $i = 0, 1, 2, 3, \ldots$ generates the sequence 1, 11, 46, 130, 295, 581, 1036, 1716, 2685, ... Using the method of finite difference, $S(2i) = \frac{1}{6}i(i+1)(2i^2 + 2i - 1)$, which can be proven easily with induction, with the induction step of $S(2(i+1)) = S(2i) + \sum_{j=1}^{2i+1} \frac{j(j+1)}{2} = \frac{1}{6}i(i+1)(2i^2 + 2i - 1) + \frac{1}{2}(\sum_{j=1}^{2i+1} j^2 + \sum_{j=1}^{2i+1} j) = \frac{1}{6}(i+1)(i+2)(2i^2 + 6i + 3) = \frac{1}{6}(i+1)[(i+1)+1][2(i+1)^2 + 2i - 1]$.

For n odd, define $S(1) = 0$, $S(2(i+1)+1) = S(2i+1) + \sum_{j=1}^{2i+2} \frac{j(j+1)}{2}$ for $i = 0, 1, 2, 3, \ldots$ The sequence generated is 4, 24, 80, 200, 420, 784, 1344, 2160, 3300, ... Similarly, $S(2i+1) = \frac{1}{3}i(i+1)^2(i+2)$, which again can be proven by induction.

Therefore, the number of rectangles bounded by a n-staircase is

$$2S(n) = \begin{cases} \frac{1}{3}i(i+1)(2i^2 + 2i - 1), & \text{for } \; n = 2i; \\ \\ \frac{2}{3}i(i+1)^2(i+2), & \text{for } \; n = 2i+1. \end{cases}$$

with $i = 1, 2, 3, \ldots$

■

Addendum.

1. $S(n) = 8 \times \binom{n+2}{4} + \binom{n+1}{2}$ generates the sequence 1, 11, 46, 130, 295, 581, 1036, 1716, 2685, ... for $n = 1, 2, 3, \ldots$
2. $S(n) = 8 \times \binom{n+2}{4} + 4 \times \binom{n+2}{3}$ generates 4, 24, 80, 200, 420, 784, 1344, 2160, 3300, ... for $n = 1, 2, 3, \ldots$

3. The generating function for the sequence 1, 4, 11, 24, 46, 80, 130, 200, 295, 420, 581, 784, 1036, 1344, 1716, 2160, ... is $\frac{1}{(1+x)(1-x)^5}$.