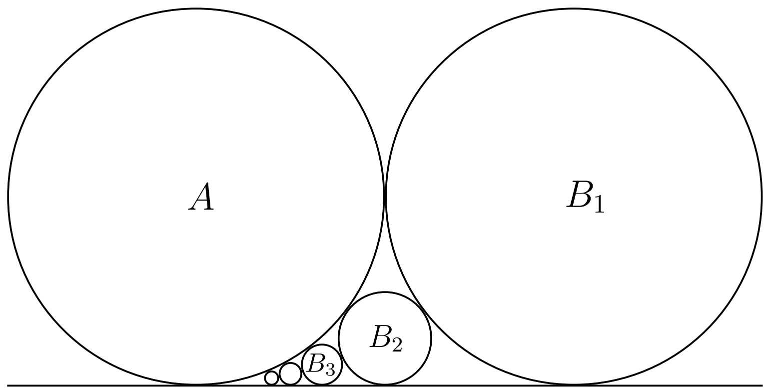
1086. *Proposed by Nathan Bronson, Chapel Hill, NC.*

A and B_1 are tangent circles of radius 1, both tangent to a line as shown. B_{i+1} is a circle tangent to A, B_i, and the line for all $i \geq 1$. The interior of no circles overlap. What is the sum of the areas of all B_i, $i \geq 1$?



Solution by **Rex H. Wu**, *Brooklyn, NY.*

Let's determine the coordinates where the centers of the circles B_i fall on. In figure 1, point $A(0,1)$ is the center of circle A and point $B(x,y)$ is the center of any circle B as stated in the question. Connect points A and B. Drop the perpendicular lines from B to the x-axis and the y-axis.

In right $\triangle ABC$, we have $(1+y)^2 = (1-y)^2 + x^2$, or $y = x^2/4$. In other words, the coordinates of the centers of the circles B_i are in the form $(x_i, \frac{x_i^2}{4})$.

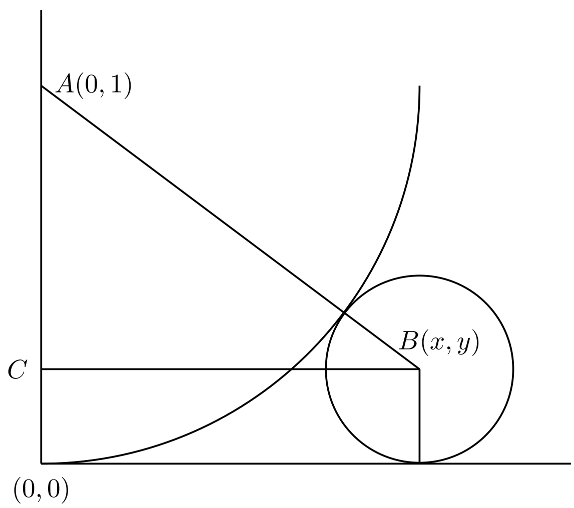


Figure 1

Next, look at figure 2. Circles B_i and B_{i+1} with centers $B_i(x_i, \frac{x_i^2}{4})$ and $B_{i+1}(x_{i+1}, \frac{x_{i+1}^2}{4})$are two circles tangent to each other. Again, drop perpendicular lines from B_i and B_{i+1} to the x-axis. Draw line $B_{i+1}C$ so it is parallel to the x-axis. Connect the centers B_i and B_{i+1}.

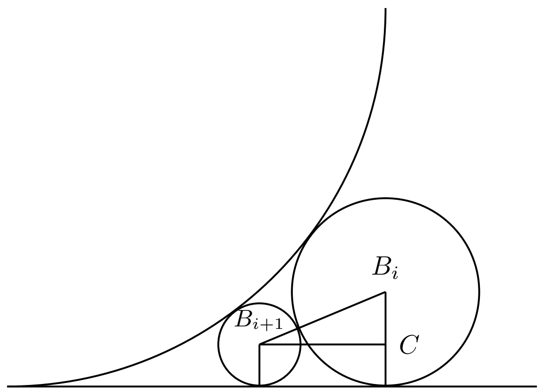


Figure 2

In right $\triangle B_i B_{i+1} C$, we have

$$\begin{aligned} \left(\frac{x_i^2}{4} - \frac{x_{i+1}^2}{4}\right)^2 + (x_i - x_{i+1})^2 &= \left(\frac{x_i^2}{4} + \frac{x_{i+1}^2}{4}\right)^2 \\ (2x_i - 2x_{i+1} + x_i x_{i+1})(2x_i - 2x_{i+1} - x_i x_{i+1}) &= 0 \end{aligned}$$

We assume circle B_i has a larger radius than circle B_{i+1}. Then $(2x_i - 2x_{i+1} + x_i x_{i+1}) > 0$. We can only have $2x_i - 2x_{i+1} - x_i x_{i+1} = 0$, or

$$x_{i+1} = \frac{2x_i}{2 + x_i}$$

From the previous conclusion, we have

$$y_{i+1} = \frac{x_{i+1}^2}{4} = \left(\frac{x_i}{2 + x_i}\right)^2$$

If we place the circles in a coordinate system as we do in figure 1, then the coordinates of the center B_1 is $(2, 1)$, B_2 is $(1, \frac{1}{4})$, B_3 is $(\frac{2}{3}, \frac{1}{9})$, etc. In general, x_n has coordinates $(\frac{2}{n}, \frac{1}{n^2})$. This can be easily established by using induction from the last two equations.

Since y_i is the radius of circle B_i, the area of the circles B_i, for $i \geq 1$ is

$$\pi \sum_{i=1}^{\infty} \left(\frac{1}{i^2}\right)^2 = \pi\zeta(4) = \frac{\pi^5}{90}$$

■