1088. *Proposed by Ben Klein, Davidson, NC.*

Let L be the line segment from $(0, 1/2)$ to $(1, 1)$. Fix a positive integer n and let

- $x_k = k/n$ for $0 \le k \le n$,
- P_k be the point on L with x-coordinate x_k,
- R_k be the region bounded above by L and below by the unique parabola that passes through the points P_k, P_{k+1}, and $M_k = ((x_k + x_{k+1})/2, 0)$ [1], and
- A_n be the sum of the areas of the R_k's.

  Find $\lim_{n\to\infty} A_n$.

Solution by **Rex H. Wu**, *Brooklyn, NY.*

The line L is $L = x/2 + 1/2$.

The unique parabola that goes through the three points $P_k(k/n, k/2n+1/2)$, $P_{k+1}((k+1)/n, (k+1)/2n + 1/2)$ and $M_k((2k+1)/2n, 0)$ is given by

$$\det \begin{vmatrix} x^2 & x & y & 1 \\ \left(\frac{k}{n}\right)^2 & \frac{k}{n} & \frac{k}{2n} + \frac{1}{2} & 1 \\ \left(\frac{k+1}{n}\right)^2 & \frac{k+1}{n} & \frac{k+1}{2n} + \frac{1}{2} & 1 \\ \left(\frac{2k+1}{2n}\right)^2 & \frac{2k+1}{2n} & 0 & 1 \end{vmatrix} = 0.$$

The above determinant gives

$$y = n(2n+2k+1)x^2 - (4kn+2n+4k+4k^2+\frac{1}{2})x + (\frac{2k^3+3k^2+k}{n} + 2k^2 + 2k + \frac{1}{2})$$

Let

$$\begin{aligned} f(x) &= L - y \\ &= -n(2n+2k+1)x^2 + (2k+1)(2n+2k+1)x - \frac{k(k+1)(2n+2k+1)}{n} \\ &= (2n+2k+1)[-nx^2 + (2k+1)x - \frac{k(k+1)}{n}] \end{aligned}$$

Then the area R_k bounded by the line segment L and the parabola y in the

[1] Note that I changed the interval for R_k, which makes it easier to manipulate.

interval $[k/x,(k+1)/x]$ is

$$\begin{aligned} R_k &= \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(x)dx \\ &= (2n+2k+1)\int_{\frac{k}{n}}^{\frac{k+1}{n}} \left[-nx^2+(2k+1)x-\frac{k(k+1)}{n}\right]dx \\ &= (2n+2k+1)\left[-\frac{nx^3}{3}+\frac{(2k+1)x^2}{2}-\frac{k(k+1)x}{n}\right]\Bigg|_{\frac{k}{n}}^{\frac{k+1}{n}} \\ &= \frac{1}{6n^2}+\frac{k}{3n^2}+\frac{1}{3n}. \end{aligned}$$

By definition,

$$\begin{aligned} A_n &= \sum_{k=0}^{n-1} R_k \\ &= \sum_{k=0}^{n-1}\left[\frac{1}{6n^2}+\frac{k}{3n^2}+\frac{1}{3n}\right] \\ &= \frac{1}{2}. \end{aligned}$$

Then it is obvious that $\lim_{n\to\infty} A_n = 1/2$.

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