

1090. Proposed by Robert C. Gebhardt, Hopatcong, NJ.
Find the exact value of

$$\sum_{n=1}^{\infty} \frac{n^2(n+2)^2}{(n+1)^6} = \frac{9}{64} + \frac{64}{729} + \frac{225}{4096} + \cdots$$

Solution by Rex H. Wu, Brooklyn, NY.

Observe that $n^2(n+2)^2 = (n+1)^4 - 2(n+1)^2 + 1$. Then

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n^2(n+2)^2}{(n+1)^6} &= \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} - 2 \sum_{n=1}^{\infty} \frac{1}{(n+1)^4} + \sum_{n=1}^{\infty} \frac{1}{(n+1)^6} \\ &= [\zeta(2) - 1] - 2[\zeta(4) - 1] + [\zeta(6) - 1] \\ &= \frac{\pi^2}{6} - \frac{2\pi^4}{90} + \frac{\pi^6}{945} \\ &= \frac{1575\pi^2 - 210\pi^4 + 10\pi^6}{9450} \end{aligned}$$

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