1113. *Proposed by Robert C. Gebhardt, Hopatcong, NJ*

Find a closed form for each of the sums below, where the constant $k > 1$ (not necessarily an integer): $\sum_{n=1}^{\infty} \frac{n^p}{k^n}$ for $p = 1, 2, 3, 4$ and 5.

Solution by **Rex H. Wu**, *Brooklyn, NY.*

The solution is obvious if we know the generating functions for the series sought in the problem. For $x < 1$,

$$\begin{aligned}
\sum_{n=1}^{\infty} nx^n &= \frac{x}{(1-x)^2} \\
\sum_{n=1}^{\infty} n^2x^n &= \frac{x^2+x}{(1-x)^3} \\
\sum_{n=1}^{\infty} n^3x^n &= \frac{x^3+4x^2+x}{(1-x)^4} \\
\sum_{n=1}^{\infty} n^4x^n &= \frac{x^4+11x^3+11x^2+x}{(1-x)^4} \\
\sum_{n=1}^{\infty} n^5x^n &= \frac{x^5+26x^4+66x^3+26x^2+x}{(1-x)^5}
\end{aligned}$$

For $k > 1$, let $x = \frac{1}{k}$. Then

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n}{k^n} &= \frac{\frac{1}{k}}{(1-\frac{1}{k})^2} \\
&= \frac{k}{(k-1)^2}
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n^2}{k^n} &= \frac{\frac{1}{k^2}+\frac{1}{k}}{(1-\frac{1}{k})^3} \\
&= \frac{k(k+1)}{(k-1)^3}
\end{aligned}$$

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n^3}{k^n} &= \frac{\frac{1}{k^3}+\frac{4}{k^2}+\frac{1}{k}}{(1-\frac{1}{k})^4} \\
&= \frac{k(k^2+4k+1)}{(k-1)^4}
\end{aligned}$$

$$\sum_{n=1}^{\infty} \frac{n^4}{k^n} = \frac{\frac{1}{k^4}+\frac{11}{k^3}+\frac{11}{k^2}+\frac{1}{k}}{(1-\frac{1}{k})^5}$$

$$= \frac{k(k+1)(k^2+10k+1)}{(k-1)^5}$$

$$\sum_{n=1}^{\infty} \frac{n^5}{k^n} = \frac{\frac{1}{k^5}+\frac{26}{k^4}+\frac{66}{k^3}+\frac{26}{k^2}+\frac{1}{k}}{(1-\frac{1}{k})^6}$$

$$= \frac{k(k^4+26k^3+66k^2+26k+1)}{(k-1)^5}$$

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