1125. *Proposed by David Wells, Penn State New Kensington, Upper Burrell, PA*

For each positive integer n, let $P(n)$ be the product of the decimal digits of n, let $P_1(n) = P(n)$, and for $k \geq 2$, let $P_k(n) = P(P_{k-1}(n))$. Prove that $P_k(n) = 1$ for some k if and only if n contains no digits other than 1.

Solution by **Rex H. Wu**, *Brooklyn, NY.*

If $n = 111 \cdots 111$, then $P_1(n) = 1$.

If $P_k(n) = 1$, we need to show $n = 111 \cdots 111$.

If any number n contains the digits $0, 2, 4, 6$ or 8, $P_k(n)$ will eventually end up with an even number. Therefore, n must not contain any even digits. If n contains a 5, $P_k(n)$ will end with 0 or 5. 5 is excluded.

That means we only need to look at the number n consists of 1, 3, 7 and 9.

Suppose there is such an n, then for all $P_i(n)$, $k \geq i \geq 1$, the digits of $P_i(n)$ can only consist of 1, 3, 7 or 9. Besides, $P_i(n)$ can only contain 3 or 7 as prime factors because $P_{i+1}(n)$ is the product of the digits of $P_i(n)$.

In order for $P_k(n) = 1$, $P_{k-1}(n) = 111 \cdots 111$ because the only factor of $P_k(n) = 1$ is 1. The question is can there exist a $P_{k-2}(n)$ such that it contains 3, 7 or 9.

Suppose $P_{k-2}(n)$ contains a digit 3. Then 3 divides $P_{k-1}(n) = 111 \cdots 111$. This implies the number of 1's in $P_{k-1}(n)$ is a multiple of 3 (e.g. 111, 111111, 111111111, etc).

Suppose $P_{k-2}(n)$ contains a digit 7, then $P_{k-1}(n)$ is divisible by 7.

We will apply the divisibility rule for 7 here. For any number, multiply each digit beginning on the right hand side (ones) by 1, 3, 2, 6, 4, 5. Repeat this sequence as necessary. If the sum of these products is divisible by 7, then the original number is divisible by 7.

$P_{k-1}(n)$ is divisible by 7 only if the number of 1's in $P_{k-1}(n)$ is a multiple of 6 (e.g., 111111, 111111111111, etc.) since the only way to add $1, 3, 2, 6, 4, 5$ in that order and obtain a multiple of 7 is to add all 6 of them.

Combining these results, all we need to look at is the number 111. Any number of the form $111 \cdots 111$ divisible by 3 or 7 will be divisible by 111. $111 = 3 \times 37$. Thus, any number of the form $111 \cdots 111$ divisible by 3 or 7 will also be divisible by 37. This contradicts the previous conclusion that $P_i(n)$ can only contain 3 or 7 as prime factors. Therefore, there cannot be a $P_{k-2}(n)$ which contains the digits 3, 7 or 9.

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