1133. *Proposed by Arthur Holshouser, Charlotte, NC; Anita Chatelain and Joe Albree, Auburn University at Montgomery, Montgomery, AL*

In Pillow Problem 14, Lewis Carroll proved in his head that 3 times the sum of 3 squares is also the sum of 4 squares. (Of course, 0^2 is considered to be a square).

1. Prove pillow problem 14.
2. Prove that $(a^2+b^2+c^2)(x^2+y^2+z^2)$ is also the sum of 4 squares.
2. Prove that $\prod_{i=1}^{n}(x_i^2+y_i^2+z_i^2+v_i^2)$ is also the sum of 4 squares by first proving that $(a^2+b^2+c^2+d^2)(x^2+y^2+z^2+v^2)$ is the sum of 4 squares.

Solution by **Rex H. Wu**, *Brooklyn, NY.*

3. The well known Euler identity does the trick here.

$$\begin{aligned}(a^2+b^2+c^2+d^2)(x^2+y^2+z^2+v^2) &= \\ (ax+by+cz+dv)^2 &+ (ay-bx+cv-dz)^2 \\ +(az-cx+dy-cv)^2 &+ (av-dx+bz-cy)^2\end{aligned}$$

Since the product of two sums of four squares is again a sum of four squares, this sum multiplied by the third sum of four squares is a sum of four squares, etc. Therefore,

$$\prod_{i=1}^{n}(x_i^2+y_i^2+z_i^2+v_i^2) = A^2+B^2+C^2+D^2$$

for some integer A, B, C and D.

2. Let $d=0$ and $v=0$ in the Euler identity, we have

$$\begin{aligned}(a^2+b^2+c^2)(x^2+y^2+z^2) &= (ax+by+cz)^2 \\ &+ (ay-bx)^2+(az-cx)^2+(bz-cy)^2\end{aligned}$$

1. Using the result in part 2, let $x=y=z=1$.

$$3(a^2+b^2+c^2) = (a+b+c)^2+(a-b)^2+(a-c)^2+(b-a)^2$$

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