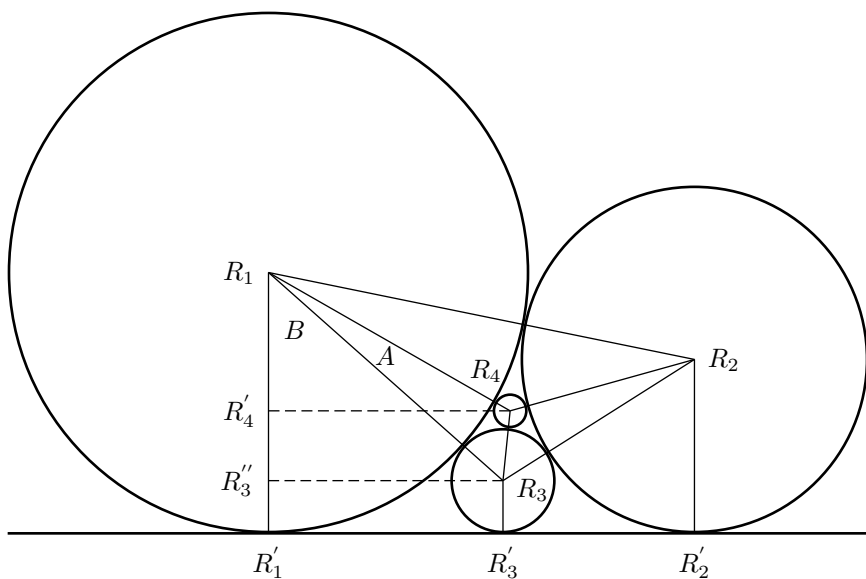


1136. Proposed by Stas Molchanov, University of North Carolina, Charlotte
 Four planar circles are pair-wise externally tangent. Three of the circles are also tangent to a line L . If the radius of the fourth circle is one unit, what is the distance of its center from L ?

Solution by **Rex H. Wu**, Brooklyn, NY.



In the above figure, circles with centers R_1, R_2, R_3 and R_4 , radii r_1, r_2, r_3 and r_4 , respectively, are mutually tangent. The radii $R_1R'_1, R_2R'_2$ and $R_3R'_3$ are perpendicular to the line L . The lines $R_4R'_4$ and $R_3R''_3$ are parallel to the line L . Let $A = \angle R_4R_1R_3$ and $B = \angle R_3R_1R'_1$.

Descartes' circle theorem states that

$$2 \left(\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r_4^2} \right) = \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} \right)^2$$

If the fourth circle is surrounded by the other three circles, then

$$\frac{1}{r_4} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + 2\sqrt{\frac{1}{r_1r_2} + \frac{1}{r_1r_3} + \frac{1}{r_2r_3}}.$$

A straight line is considered to be a circle of infinite radius. In that case, the above degenerates to

$$\frac{1}{r_3} = \frac{1}{r_1} + \frac{1}{r_2} + 2\sqrt{\frac{1}{r_1r_2}}.$$

In our problem, suppose $r_1 \geq r_2$ and $r_1 = \alpha r_2$, for some real number α .
Then

$$\frac{1}{r_3} = \frac{1}{\alpha r_2} + \frac{1}{r_2} + 2\sqrt{\frac{1}{\alpha r_2^2}} = \frac{1 + \alpha + 2\sqrt{\alpha}}{\alpha r_2}.$$

$$\begin{aligned} \frac{1}{r_4} &= \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + 2\sqrt{\frac{1}{r_1 r_2} + \frac{1}{r_1 r_3} + \frac{1}{r_2 r_3}} \\ &= \frac{1}{\alpha r_2} + \frac{1}{r_2} + \frac{1 + \alpha + 2\sqrt{\alpha}}{\alpha r_2} + 2\sqrt{\frac{1}{\alpha r_2^2} + \frac{1 + \alpha + 2\sqrt{\alpha}}{\alpha^2 r_2^2} + \frac{1 + \alpha + 2\sqrt{\alpha}}{\alpha r_2^2}} \\ &= \frac{2 + 2\alpha + 2\sqrt{\alpha} + 2\sqrt{1 + 3\alpha + \alpha^2 + 2\sqrt{\alpha} + 2\alpha\sqrt{\alpha}}}{\alpha r_2} \end{aligned}$$

Noticing that $(1 + \alpha + \sqrt{\alpha})^2 = 1 + 3\alpha + \alpha^2 + 2\sqrt{\alpha} + 2\alpha\sqrt{\alpha}$, which is the radicand in the last line, the above becomes

$$\frac{1}{r_4} = \frac{4(1 + \alpha + \sqrt{\alpha})}{\alpha r_2}$$

In terms of r_4 , we have

$$r_2 = \frac{4r_4(1 + \alpha + \sqrt{\alpha})}{\alpha}$$

Just a recapitulation, for $r_4 = 1$,

$$r_1 = \alpha r_2 = 4(1 + \alpha + \sqrt{\alpha})$$

$$r_2 = \frac{4(1 + \alpha + \sqrt{\alpha})}{\alpha}$$

$$r_3 = \frac{4(1 + \alpha + \sqrt{\alpha})}{1 + \alpha + 2\sqrt{\alpha}}$$

$$r_4 = 1$$

In $\triangle R_1 R_4 R_3$, using the law of cosine,

$$\cos A = \frac{(r_1 + 1)^2 + (r_1 + r_3)^2 - (r_3 + 1)^2}{2(r_1 + 1)(r_1 + r_3)}$$

In $\triangle R_1 R_3 R_3''$, $R_1 R_3'' = r_1 - r_3$.

Then,

$$A = \cos^{-1} \left(\frac{(r_1 + 1)^2 + (r_1 + r_3)^2 - (r_3 + 1)^2}{2(r_1 + 1)(r_1 + r_3)} \right)$$

$$B = \cos^{-1} \left(\frac{r_1 - r_3}{r_1 + r_3} \right)$$

From the above, through Pythagorean's theorem, the following can be obtained:

$$\begin{aligned}
\cos A &= \frac{(r_1 + 1)^2 + (r_1 + r_3)^2 - (r_3 + 1)^2}{2(r_1 + 1)(r_1 + r_3)} \\
\sin A &= \frac{4\sqrt{r_1 r_3}(1 + r_1 + r_3)}{2(r_1 + 1)(r_1 + r_3)} \\
\cos B &= \frac{r_1 - r_3}{r_1 + r_3} \\
\sin B &= \frac{2\sqrt{r_1 r_3}}{r_1 + r_3} \\
\cos(A + B) &= \cos A \cos B - \sin A \sin B \\
&= \frac{(r_1 - r_3)[r_1 - r_3 + r_1(r_1 + r_3)] - 4r_1 r_3 \sqrt{1 + r_1 + r_3}}{(r_1 + 1)(r_1 + r_3)^2}
\end{aligned}$$

Substitute $r_1 = 4(1 + \alpha + \sqrt{\alpha})$ and $r_3 = \frac{4(1 + \alpha + \sqrt{\alpha})}{1 + \alpha + 2\sqrt{\alpha}}$ into the last expression. It turns out the radicand $(1 + r_1 + r_3)$ is a perfect square in terms of α , $(1 + r_1 + r_3) = \left(\frac{3 + 2\alpha + 3\sqrt{\alpha}}{1 + \sqrt{\alpha}}\right)^2$. With tedious algebraic manipulations, the final result turns out to be:

$$\cos(A + B) = \frac{4\alpha + 4\sqrt{\alpha} - 3}{4\alpha + 4\sqrt{\alpha} + 5}$$

The distance from R_4 to the line L is the same as $R'_4 R'_1$.

$$\begin{aligned}
R'_4 R'_1 &= R_1 R'_1 - R_1 R'_4 \\
&= r_1 - (r_1 + r_4) \cos(A + B) \\
&= r_1 - (r_1 + 1) \cos(A + B)
\end{aligned}$$

Again, substitute the previous results for r_1 and $\cos(A + B)$ into this expression gives the final answer, $R'_4 R'_1 = 7$. ■