1137. *Proposed by Peter Lindstrom, Batavia, NY*
Let n be a positive integer and T_i be the i^{th} triangular number.
Find the value of

$$\lim_{k\to\infty}\sum_{i=1}^{k}\frac{\sum_{j=1}^{n}\binom{n}{j}i^{n-j}}{(T_i)^n}.$$

Solution by **Rex H. Wu**, *Brooklyn, NY.*
The numerator is the binomial expansion without the first term:

$$(i+1)^n = \sum_{j=0}^{n}\binom{n}{j}i^{n-j}$$

$$\sum_{j=1}^{n}\binom{n}{j}i^{n-j} = (i+1)^n - i^n$$

Then the fractional part becomes:

$$\begin{aligned}\frac{\sum_{j=1}^{n}\binom{n}{j}i^{n-j}}{(T_i)^n} &= \frac{(i+1)^n - i^n}{\left(\frac{i(i+1)}{2}\right)^n}\\ &= \frac{(i+1)^n}{\frac{i^n(i+1)^n}{2^n}} - \frac{i^n}{\frac{i^n(i+1)^n}{2^n}}\\ &= \left(\frac{2}{i}\right)^n - \left(\frac{2}{i+1}\right)^n\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{k\to\infty}\sum_{i=1}^{k}\frac{\sum_{j=1}^{n}\binom{n}{j}i^{n-j}}{(T_i)^n} &= \lim_{k\to\infty}\sum_{i=1}^{k}\left[\left(\frac{2}{i}\right)^n - \left(\frac{2}{i+1}\right)^n\right]\\ &= \lim_{k\to\infty}\left[2^n - \left(\frac{2}{1+1}\right)^n + \left(\frac{2}{2}\right)^n - \left(\frac{2}{2+1}\right)^n\right.\\ &\quad\left. + \left(\frac{2}{3}\right)^n - \left(\frac{2}{3+1}\right)^n + \cdots - \left(\frac{2}{k+1}\right)^n\right]\\ &= \lim_{k\to\infty}\left[2^n - \left(\frac{2}{k+1}\right)^n\right]\\ &= 2^n\end{aligned}$$

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