Rex H. Wu
Brooklyn, NY

1603 Solution:

Solve $x^5 + y^5 = (x + y)^3$ for integer solutions.

Some obvious solutions are $(x, y) = (0, 0), (0, 1), (0, -1), (1, 0), (-1, 0), (n, -n)$ and $(-n, n)$ where n is a positive integer.

Now let's assume $x+y \neq 0$ and factor x^5+y^5, we have

$$(x+y)(x^4-x^3y+x^2y^2-xy^3+y^4) = (x + y)^3$$

$$x^4-x^3y+x^2y^2-xy^3+y^4 = (x + y)^2$$

The above can be rearranged into

$$(x + y)^2 (xy + 1) = (x^2 + y^2)^2 + x^2 y^2$$

The right hand side is always non-negative, which implies $(xy+1)$ on the left and side cannot be negative. Therefore, x and y cannot be of the opposite sign.

By symmetry, we know if (x_0, y_0) is a solution, then $(-x_0, -y_0)$ is also a solution to the original equation.

Let's assume $y = (m/n)x$, for some positive integers m and n and $x > 0$. Since the functions $f(x,y) = x^5+y^5$ and $g(x,y) = (x + y)^3$ are symmetric about the plane $x = y$, we can consider the case where $m \geq n$.

Substitute $y = (m/n)x$ into the original equation and get

$$x^5 + [(m/n)x]^5 = [x + (m/n)x]^3$$

or

$$x^2 = [1+(m/n)]^3/[1+(m/n)^5] \qquad (*)$$

There are three situations to consider here. Namely, (i) $m/n \geq 2$, (ii) $2 > m/n > 1$ and (iii) $m/n = 1$.

In case (i), we have $[1+(m/n)^5]>[1+(m/n)]^3$. Then x^2 becomes a fraction and then original equation cannot have an integer solution.

In case (ii), let's change the notation a little. Let $m/n = 1 + 1/q$, for some rational number $q > 1$. Then (*) becomes

$$x^2 = [1+(1+1/q)]^3/[1+(1+1/q)^5]$$

$$x^2 = (8q^5+12q^4+6q^3+q^2)/(2q^5+5q^4+10q^3+10q^2+5q+1)$$

$$x^2 = 4 - (8q^4+34q^3+39q^2+20q+4)/(2q^5+5q^4+10q^3+10q^2+5q+1)$$

This shows $x^2 < 4$. The only non-zero perfect square less than 4 is 1. Now going back to (*) and let

$$1 = [1+(m/n)]^3/[1+(m/n)^5]$$

Rearrange the above equality to get

$$3n^2 + 3nm^2 + m^2 = (m^2/n)^2$$

The left hand side of the above equality is an integer while the right hand side is an integer only if n divides m which contradicts our assuption that $2 > m/n > 1$.

In the last case, we have $m/n = 1$, then $x^2 = 4$, or $(x, y) = (2, 2)$ and $(-2, -2)$.

Therefore, the only integer solutions to the equation $x^5 + y^5 = (x + y)^3$ are $(x, y) = (0, 0)$, $(0, 1)$, $(0, -1)$, $(1, 0)$, $(-1, 0)$, $(2, 2)$, $(2, -2)$, $(n, -n)$ and $(-n, n)$, where n is a positive integer.