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In $\triangle ABC$, let BE and CD be cevians meeting the sides AC and AB at points E and D respectively. We have $AE/EC = AD/DB$. Connect points D and E. Then $DE \parallel BC$.

Look at the quadrilateral $DECB$, since $DE \parallel BC$, it is a trapezoid. We also know $BE \cong CD$. Therefore trapezoid $DECB$ is isosceles with $BD \cong CE$. It follows that $AD \cong AE$ and therefore $AB \cong AC$.

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The triangle is isosceles if such a pair of cevians divide the angles into equal proportion.

Theorem. In $\triangle ABC$, cevians BE and CD are congruent. If $\frac{\angle ABE}{\angle EBC} = \frac{\angle ACD}{\angle DCB}$, then $AB \cong AC$.

Proof. Draw segment DF such that $DF \cong BE$ and $DF \parallel BE$. Connect FE and FC.

Quadrilateral $BDFE$ then is a parallelogram, with $\angle DFE \cong \angle DBE$. We also have $DF \cong BE \cong CD$ so that $\triangle DFC$ is isosceles with $\angle DFC \cong \angle DCF$.

I will use the following proposition which I won't prove here.

Proposition. In $\triangle ABC$ and $\triangle A'B'C'$, $AC \cong A'C'$ and $BC \cong B'C'$. If $\angle ACB > \angle A'C'B'$, then $AB > A'B'$.

Now, suppose $\angle EBC > \angle DCB$, then by the above proposition, $EC > DB$ or $EC > EF$ which in turn implies $\angle EFC > \angle ECF$.

Since $\frac{\angle ABE}{\angle EBC} = \frac{\angle ACD}{\angle DCB}$ and $\angle EBC > \angle DCB$, we have $\angle ABE > \angle ACD$. $\angle DFE = \angle ABE$ implies $\angle DFE > \angle DCE$.

Finally we have $\angle DFC = \angle DFE + \angle EFC > \angle DCE + \angle ECF = \angle DCF$, which contradicts $\angle DFC = \angle DCF$.

Suppose on the other hand, $\angle EBC < \angle DCB$. By the same reasoning, we end up with $\angle DFC < \angle DCF$, again a contradiction.

Therefore, $\angle EBC = \angle DCB$. Then $\angle ABE = \angle ACD$ and finally $\angle ABC = \angle ACB$.

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