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December 8, 2001

Mathematics Magazine Problem 1637.

In the circle $x^2 + y^2 = 1$, let's pick reference points $A(1, 0)$ and $B(\frac{a}{c}, \frac{b}{c})$, where a, b and $c \in \mathbb{Z}$. Then we have $\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$, or $a^2 + b^2 = c^2$.

The solutions to $a^2 + b^2 = c^2$ are the Pythagorean triples $(a, b, c) = (p^2 - q^2, 2pq, p^2 + q^2)$ with p and $q \in \mathbb{Z}$ and of the opposite parity.

The distance between points A and B is $\|AB\| = \sqrt{\left(\frac{a}{c} - 1\right)^2 + \left(\frac{b}{c}\right)^2} = \sqrt{\left(\frac{p^2 - q^2}{p^2 + q^2} - 1\right)^2 + \left(\frac{2pq}{p^2 + q^2}\right)^2} = \frac{2q}{p^2 + q^2}\sqrt{p^2 + q^2}$.

It is readily seen that the distance $\|AB\|$ is irrational if $p^2 + q^2$ is not a perfect square. There are plenty of such numbers. For example, $(p, q) = (1, q)$ where $q \in \{2\mathbb{Z}\}$ is one. $(p, q) = (2, q)$ where $q \in \{2\mathbb{Z} + 1\}$ provides another. ■