Rex H. Wu
Brooklyn, NY

Proposal Following Problem 697, let $L(n) = m$ if m is the smallest positive integer such that n divides $m!$. For m a positive integer, define $P(m) = \{n : L(n) = m\}$.

(a) Show that the cardinality of $P(m)$, $|P(m)|$, can never be a prime.

(b) Show that for any prime p, there is a m such that p divides $|P(m)|$.

PROOF: We know x_0 is a solution to $L(x) = m$ if and only if $x_0|m!$ and $m \nmid (m!/x_0)$. In other words, the solutions are factors of $m!$ that are not divisible by m. Therefore, $|P(m)| = \tau(m!) - \tau((m-1)!)$, where $\tau(n)$ is the number of factors of n, i.e. if $n = p_0^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, then $\tau(n) = (\alpha_0 + 1)$ $(\alpha_1 + 1)$ $(\alpha_2 + 1) \cdots (\alpha_n + 1)$.

This can be easily seen if we look for the number of factors of $m!$ that are not divisible by m. To look for those, we will find out the number of factors that are divisible by m, i.e., factors of the form mA, for some integer A. Since $mA|m!$, we have $A|(m-1)!$. There is a total of $\tau((m-1)!)$ such A's. Since there are $\tau(m!)$ factors of $m!$, there are $|P(m)| = \tau(m!) - \tau((m-1)!)$ factors of $m!$ that are not divisible by k.

(a) Once we know this, we can see for $m \geq 2$, $\tau(m!)$ is always even. By Bertrand's Postulate, there is always a prime, $p \in [m/2, m]$ if m is even or $p \in [(m+1)/2, (m+1)]$ if m is odd. Furthermore, the exponent of this p is 1. Then $\tau(m!) = 2B$ for some integer B. It follows immediately that $|P(m)|$ is even if $m \geq 3$. It remains to check for $|P(1)|$ and $|P(2)|$, which turn out to be 1 in both cases.

(b) For any prime p, take m such that $p(p-1) < m < p^2$. Then $m = p^{p-1}M$ and $m - 1 = p^{p-1}N$, for some integers M and N. Written in canonical prime factor forms, $M = p_0^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_i^{\alpha_i}$ and $N = p_0^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_j^{\alpha_j}$. Then $|P(m)| = \tau(m!) - \tau((m-1)!) = p(\alpha_0+1)(\alpha_1+1) \cdots (\alpha_i+1) - p(\alpha_0+1)(\alpha_1+1) \cdots (\alpha_j+1) = p((\alpha_0+1)(\alpha_1+1) \cdots (\alpha_i+1) - (\alpha_0+1)(\alpha_1+1) \cdots (\alpha_j+1))$, which is divisible by p.

QED