SOME PROPERTIES ON THE EQUATION S(X) = K

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In 1979, Florentin Smarandache introduced a number theoretic function. For any positive integer n, the Smarandache function $\mathrm{S}(n)$ is defined as $\mathrm{S}(n) = k$ if k is the smallest positive integer such that n divides $k!$. Since then, some interesting properties have been discovered about this function. Just one example, for $x > 4$, the expression

$$\pi(x) = -1 + \sum_{k=2}^{x} \left[\frac{\mathrm{S}(k)}{k} \right],$$

where $[x]$ is the greatest integer function, gives the exact number of primes less than or equal to x, [1].

In this note, we will look at some elementary properties associated with the equation $\mathrm{S}(x) = k$.

First, let's see how we can solve the equation $\mathrm{S}(x) = k$. Suppose

$$k = p_1^{\beta_1} p_2^{\beta_2} p_3^{\beta_3} \cdots p_j^{\beta_j} \text{ and}$$
$$s(k-1)! = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_j^{\alpha_j} p_{j+1}^{\alpha_{j+1}} \cdots p_t^{\alpha_t}.$$

Then $k! = p_1^{\alpha_1+\beta_1} p_2^{\alpha_2+\beta_2} p_3^{\alpha_3+\beta_3} \cdots p_j^{\alpha_j+\beta_j}\ p_{j+1}^{\alpha_{j+1}} \cdots p_t^{\alpha_t}$, for some prime p_i and nonnegative integers α_i and β_i; with $i = 1, 2, 3, \ldots, j, \ldots, t$. Here, j is used as the number of prime factors of k. Note that if p_i is a prime that divides k but not $(k-1)!$, then take $\alpha_i = 0$. If x_0 were a solution to $\mathrm{S}(x) = k$, then $x_0 \mid k!$. Furthermore, $x_0 \nmid (k-1)!$. Obviously x_0 contains some factor $p_i^{\gamma_i}$, where $\alpha_i < \gamma_i \leq \alpha_i + \beta_i$, for some $i = 1, 2, 3, \ldots, j$. So we have our first conclusion.

THEOREM 1. *x_0 is a solution to $\mathrm{S}(x) = k$ if and only if $x_0 = MNQ$, where*

$$M = \prod_{i \in I} p_i^{\lambda_i},$$

where I can be any nonempty subset of $\{1, 2, 3, \ldots, j\}$ and $1 \leq \lambda_i \leq \beta_i$;

$$N = \prod_{i \in I} p_i^{\alpha_i},$$

where, again, if p_i is a prime that divides k but not $(k-1)!$, then take $\alpha_i = 0$; and Q is any factor of $(k-1)!/N$.

Proof. We have

$$MN = \prod_{i \in I} p_i^{\alpha_i + \lambda_i}.$$

Since $\alpha_i < \alpha_i + \lambda_i \leq \alpha_i + \beta_i$, we know $MN \mid k!$ but $MN \nmid (k-1)!$. For N, using the highest exponent α_i so that $p_i^{\alpha_i} \mid (k-1)!$ is essential. Otherwise, MN may divide $(k-1)!$ and rendering MN not a solution. Furthermore, if Q divides $(k-1)!/N$, then

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MNQ divides $M(k-1)!$ which in turn divides $k!$. Therefore, MNQ is a solution to $\mathrm{S}(x) = k$.

Observe that using all the nonempty subsets of $\{ 1, 2, 3, \ldots, j \}$ for M would generate all the factors of k. In combination with all the factors of $(k-1)!/N$, we have all the solutions to $\mathrm{S}(x) = k$.

Suppose there is a solution x_0 to $\mathrm{S}(x) = k$, we are going to show x_0 is of the form MNQ. Let $x_0{=}p_1^{\varepsilon_1} p_2^{\varepsilon_2} p_3^{\varepsilon_3} \cdots p_m^{\varepsilon_m}$. Note that $\mathrm{S}(x_0) = k = \max\{\mathrm{S}(p_i^{\varepsilon_i})\}$ for $i = 1, 2, 3, \ldots, m$; with $m \le t$. Obviously, if p_i is not a factor of k, $\mathrm{S}(p_i^{\varepsilon_i}) < k$. Even if p_i is a factor of k, if $\varepsilon_i \le \alpha_i$, then $\mathrm{S}(p_i^{\varepsilon_i}) < k$ because $\mathrm{S}(p_i^{\varepsilon_i}) \mid (k-1)!$. If $\varepsilon_i > \alpha_i + \beta_i$, $\mathrm{S}(p_i^{\varepsilon_i}) > k$. Therefore, we have $p_i \mid k$ and $\alpha_i < \varepsilon_i \le \alpha_i + \beta_i$. Notice that there can be more than one such p_i's such that $\mathrm{S}(p_i^{\varepsilon_i}) = \max\ \{\mathrm{S}(p_i^{\varepsilon_i})\}$ if k has more than one prime factor. This shows if x_0 were a solution, then x_0 contains $MN = \prod_{i \in I} p_i^{\varepsilon_i}$, for some subset I of $\{1, 2, 3, \ldots, j\}$ and $\alpha_i < \varepsilon_i \le \alpha_i + \beta_i$. Also notice that any multiples of MN, say MNQ, is a solution to $\mathrm{S}(x) = k$, provided $MNQ \mid k!$. The question is what can Q be?

Obviously, $MNQA = k!$, for some integer A. $QA = k!/MN = (k/M)((k{-}1)!/N)$. From the previous expression, Q can be any factor of $(k-1)!/N$. What if Q contains a prime factor p_q such that p_q is also a factor of k? Then p_q must have an exponent $\varepsilon_q \le \alpha_q$, in which case $p_q^{\varepsilon_q}$ is a factor of $(k-1)!/N$. Otherwise, $\mathrm{S}(p_q^{\varepsilon_q}) = k$ if $\alpha_q < \varepsilon_q \le \alpha_q + \beta_q$, but then this factor would be part of MN. Or $\mathrm{S}(p_q^{\varepsilon_q}) > k$ if $\varepsilon_q > \alpha_q + \beta_q$. Therefore, we can only have $Q \mid ((k-1)!/N)$. □

An example would best illustrate this theorem. Let's solve $\mathrm{S}(x) = 12$. Here $k = 12 = 2^2 \times 3$, $(k-1)! = 11! = 2^8 \times 3^4 \times 5^2 \times 7 \times 11$. Let's look at the number of solutions instead of each individual solution. Obviously, the number of solutions for any particular M is $\tau(Q)$, where $\tau(n)$ is the number of factors for the positive integer n. If $n = p_0^{\alpha_0} p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$, then $\tau(n) = (\alpha_0 + 1)\ (\alpha_1 + 1)\ (\alpha_2 + 1) \cdots (\alpha_n + 1)$.

M	N	$Q = 11!/N$	$\tau(Q)$ = number of solutions
2	2^8	factors of $3^4 \times 5^2 \times 7 \times 11$	60
3	3^4	factors of $2^8 \times 5^2 \times 7 \times 11$	108
2^2	2^8	factors of $3^4 \times 5^2 \times 7 \times 11$	60
2×3	$2^8 \times 3^4$	factors of $5^2 \times 7 \times 11$	12
$2^2 \times 3$	$2^8 \times 3^4$	factors of $5^2 \times 7 \times 11$	12

Adding up the last column gives a total of 252 solutions.

While the above theorem works, it gets cumbersome if k gets large. Let's explore a little bit and look for a simpler method. We will also switch our attention to look for the number of solutions rather than all the specific solutions to $\mathrm{S}(x) = k$.

THEOREM 2. *x_0 is a solution to $\mathrm{S}(x) = k$ if and only if $x_0 \mid k!$ and $k \nmid (k!/x_0)$.*

Proof. Suppose x_0 is a solution to $S(x) = k$, then by definition, $x_0 \mid k!$ and for any $n < k$, $x_0 \nmid n!$. It suffices to show the case $n = k-1$, since if $x_0 \nmid (k-1)!$ then $x_0 \nmid n!$ for any $n < k-1$. Therefore, for $n = k-1$, $x_0 \nmid n!$ implies $kx_0 \nmid k!$ or $k \nmid (k!/x_0)$.

Let's say $x_0 | k!$ but $k \nmid (k!/x_0)$. Since $k \nmid (k!/x_0)$ is equivalent to $kx_0 \nmid k!$, or $x_0 \nmid (k-1)!$. Obviously, if $x_0 \nmid (k-1)!$, then $x_0 \nmid n!$ for any $n \le k-1$. This is the very definition of the Smarandache function. Therefore, $S(x_0) = k$. □

Theorems 1 and 2 are actually equivalent. To see if MNQ is a solution or not, all we need to do is to see if k divides $k!/(MNQ)$ or not. Suppose p_m is one of the primes used in MN, i.e. $1 \le m \le j$, then $MNQ = p_m^{\alpha_m + \lambda_m} A$, for some integer A and $1 \le \lambda_m \le \beta_m$. So $k!/(MNQ) = p_m^{\beta_m - \lambda_m} B$ for some integer B. Obviously, $k \nmid (k!/(MNQ))$ because k has a factor $p_m^{\beta_m}$ and $p_m^{\beta_m} \nmid p_m^{\beta_m - \lambda_m} B$.

COROLLARY 3. *If k is prime, then there are $\tau(k!)/2$ solutions to* S$(x) = k$.

Proof. Let's pair up the divisors of $k!$ such that the product of each pair is $k!$, i.e., if $x_0 \mid k!$, then x_0 is paired up with $k!/x_0$. k is prime implies $k \nmid (k-1)!$. Then either $k \mid x_0$ or $k \mid (k!/x_0)$ but not both. If $k \nmid x_0$, then $k!/x_0$ is a solution to S$(x) = k$. Otherwise, x_0 is. This shows exactly half of the factors of $k!$ are solutions to S$(x) = k$ if k is prime. □

Once we know theorem 2, we can look for the number of solutions to S$(x) = k$ with ease. Let's denote $\omega(k)$ the number of solutions to S$(x) = k$.

COROLLARY 4. *There are $\omega(k) = \tau(k!) - \tau((k-1)!)$ solutions to* S$(x) = k$.

Proof. According to Theorem 2, this is to look for the number of factors of $k!$ that are not divisible by k.

Let's look at the factors of $k!$, in particular, we are interested in the ones that are not divisible by k. To look for those, we will find out the number of factors that are divisible by k, i.e., factors of the form kA, for some integer A. Since $kA \mid k!$, we have $A \mid (k-1)!$. There is a total of $\tau((k-1)!)$ such A's. Since there are $\tau(k!)$ factors of $k!$, there are $\omega(k) = \tau(k!) - \tau((k-1)!)$ factors that are not divisible by k. □

Corollary 2 gives another proof to corollary 1. If k is prime, then $k = p_0$ is a prime different from all the primes less than or equal to $(k-1)$. If there are $\tau((k-1)!)$ factors for $(k-1)!$, then,$\tau(k!) = \tau(k(k-1)!) = \tau(k)\tau((k-1)!) = 2\tau((k-1)!)$, which is the same as $\omega(k) = \tau(k!)/2$.

Now let's look at the first 15 values for $\omega(k)$. Note that $\omega(12)$ confirms the result we obtained using theorem 1.

k	$\tau(k!)$	$\omega(k)$
1	1	1
2	2	1
3	4	2
4	8	4
5	16	8
6	30	14
7	60	30
8	96	36
9	160	64
10	270	110
11	540	270
12	792	252
13	1584	792
14	2592	1008
15	4032	1440

Pay attention to the $\tau(k!)$'s and $\omega(k)$'s where $\omega(k) = \tau(k!)/2 = \tau((k-1)!)$. Also look at the corresponding k. A pattern seems to arise. The k's are prime except when it is 4. One may wonder if this pattern would be true for all.

Before we go onto proving the above statement, we need to utilize a function, $E(n,p)$, which gives the largest exponent of a prime p such that $p^{E(n,p)} \mid n!$.

$$E(n,p) = \sum_{i=1}^{\infty} \left[\frac{n}{p^i}\right]$$

gives the numerical value of $E(n,p)$, where $[x]$ is the greatest integer function. In particular, if $n = p^m$,

$$E(n,p) = 1 + p + p^2 + \cdots + p^{m-1} = \frac{p^m - 1}{p - 1}.$$

Also observe that

(i) if $n = Np$, for some positive integer N, then $N = \frac{n}{p} \le E(n,p)$ with equality only when $N < p$ and

(ii) if $p_1 > p_2$, then $E(n,p_1) \le E(n,p_2)$.

Lemma 5. *If $n = Qp^m$ for some prime p and some integers Q and m, then $E(n,p) = Q\,E(p^m,p) + E(Q,p)$.*

Proof.

$$\begin{aligned} E(n,p) &= \sum_{i=1}^{\infty} \left[\frac{n}{p^i}\right] \\ &= \sum_{i=1}^{\infty} \left[\frac{Qp^m}{p^i}\right] \\ &= \sum_{i=1}^{m} Qp^{m-i} + \sum_{i=m+1}^{\infty} \left[Qp^{m-i}\right] \\ &= Q(1 + p + p^2 + \cdots + p^{m-1}) + \sum_{i=1}^{\infty} \left[\frac{Q}{p^i}\right] \\ &= Q\,E(p^m,p) + E(Q,p) \end{aligned}$$

□

Theorem 6. *If there are $\tau(k!)/2$ solutions to $\mathrm{S}(x) = k$, then k is prime or $k = 4$.*

Proof. Here, $\omega(k) = \tau(k!) - \tau((k-1)!) = \tau(k!)/2$. Or equivalently, if $\tau(k!) = 2\tau((k-1)!)$ then k is prime or $k = 4$. If we could show that k is a composite number other than 4 implies $\tau(k!) \ne 2\tau((k-1)!)$ and we are done.

Again, let's write k and $(k-1)!$ in their canonical prime factorization forms, $k = p_1^{\beta_1} p_2^{\beta_2} p_3^{\beta_3} \cdots p_j^{\beta_j}$ and $(k-1)! = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_j^{\alpha_j} p_{j+1}^{\alpha_{j+1}} \cdots p_t^{\alpha_t}$. Then, $E(k,p_i) = \alpha_i + \beta_i$ and $E(k-1,p_i) = \alpha_i$.

Next, we will look at the inequality $\beta_i \le \frac{\alpha_i}{2j}$, for some positive integer $i \le j$. In particular, we are interested in whether $\tau(k!) = 2\tau((k-1)!)$ or not when $\beta_i \le \frac{\alpha_i}{2j}$ and when $\beta_i > \frac{\alpha_i}{2j}$.

If we rewrite $k = Qp_i^{\beta_i}$, then from the lemma, we have $E(k,p_i) = E(Qp_i^{\beta_i},p_i) = Q\,E(p_i^{\beta_i},p_i) + E(Q,p_i)$. Furthermore, $\alpha_i = E(k,p_i) - \beta_i = Q\,E(p_i^{\beta_i},p_i) + E(Q,p_i) - \beta_i$. Suppose $\beta_i > \alpha_i/2j$, then a substitution for α_i and some rearrangements give $2j + 1 > (Q\,E(p_i^{\beta_i},p_i) + E(Q,p_i))/\beta_i$. And finally,

$$2j + 1 > Q\,\frac{p_i^{\beta_i} - 1}{\beta_i(p_i - 1)} + \frac{E(Q,p_i)}{\beta_i}. \tag{1}$$

Case (I). k has only one prime factor, $k = p^\beta$ with $\beta > 1$.

From the assumption, we have $j = 1$ and $Q = 1$. From Equation 1 we have

$$3 > \frac{p^\beta - 1}{\beta(p - 1)}.$$

Note that $E(Q,p)/\beta = E(1,p)/\beta = 0$. There are only a few cases that this inequality is true, namely, $(p,\beta) = (2,2),(2,3)$ and $(3,2)$, corresponding to $k = 4, 8$ and 9, respectively. By assumption $k \neq 4$. It is easy to check that $\tau(8!) \neq 2\tau(7!)$ and $\tau(9!) \neq 2\tau(8!)$.

Case (II). k has 2 distinct prime factors, $k = p_1^{\beta_1} p_2^{\beta_2}$ with $\beta_i > 0$.

Here, $j = 2$ and without loss of generality, $Q = p_1^{\beta_1}$. Again, we have

$$2j + 1 = 5 > Q \, \frac{p_2^{\beta_2} - 1}{\beta_2(p_2 - 1)} + \frac{E(Q, p_2)}{\beta_2}.$$

The inequality is true if $(Q, p_2, \beta_2) = (2, 3, 2), (2, p_2, 1), (2^2, p_2, 1)$ for some prime $p_2 \geq 3$ and $(3, p_2, 1)$ for some prime $p_2 \geq 5$.

For $(Q, p_2, \beta_2) = (2, 3, 2)$, we have $k = 18$. A little computation shows that $\tau(18!) \neq 2\tau(17!)$.

For $(Q, p_2, \beta_2) = (2, p_2, 1)$, or $k = 2p_2$, if $\tau((k-1)!) = (\alpha_1+1)(\alpha_2+1)\cdots(\alpha_t+1)$, then $\tau(k!) = (\alpha_1 + 1 + 1)(\alpha_2 + 1 + 1)(\alpha_3 + 1)\cdots(\alpha_t + 1)$. If $\tau(k!) = 2\tau((k-1)!)$, we have $(\alpha_1 + 1 + 1)(\alpha_2 + 1 + 1) = 2(\alpha_1 + 1)(\alpha_2 + 1)$. Simplifying the last equation gives $2 = \alpha_1\alpha_2$. Therefore $(\alpha_1, \alpha_2) = (1, 2)$ or $(2, 1)$. From observation (ii), we know $(\alpha_1, \alpha_2) = (1,\ 2)$ is not possible. It is also impossible for $(\alpha_1, \alpha_2) = (2,\ 1)$ because $\alpha_1 = 2 \neq E(2p_2 - 1, 2)$ for any prime p_2.

The argument is identical for $(Q, p_2, \beta_2) = (3, p_2, 1)$. When we reach $(\alpha_1, \alpha_2) = (2,\ 1)$, we have $\alpha_1 = 2 = E(3p_2 - 1, 3)$. Here, we have $3p_2 - 1$=6, 7 or 8. But then there is no $p_2 > 3$ satisfying this condition.

Similarly, for $(Q, p_2, \beta_2) = (2^2, p_2, 1)$, we have $4 = (\alpha_1 - 1)\alpha_2$ after equating $\tau((2^2p_2)!) = 2\tau((2^2p_2 - 1)!)$. Solving $4 = (\alpha_1 - 1)\alpha_2$ to get $(\alpha_1, \alpha_2) = (2,\ 4)$, $(3,\ 2)$ and $(5,\ 1)$. Again, by observation (ii), $(\alpha_1, \alpha_2) = (2,\ 4)$ is impossible. $(\alpha_1, \alpha_2) = (3,\ 2)$ is also impossible because $\alpha_1 = 3 = E(2^2p_2 - 1, 2)$ implies $2^2p_2 - 1 = 4$ or 5. But there is no such a p_2. For $(\alpha_1, \alpha_2) = (5,\ 1)$, $\alpha_1 = 5 \neq E(2^2p_2 - 1, 2)$ for any p_2.

Case (III). k has 3 distinct prime factors, $k = p_1^{\beta_1} p_2^{\beta_2} p_3^{\beta_3}$ with $\beta_i > 0$.

Here, $j = 3$ and $Q = p_1^{\beta_1} p_2^{\beta_2} \geq 2 \times 3 = 6$. The inequality

$$2j + 1 = 7 > Q \, \frac{p_3^{\beta_3} - 1}{\beta_3(p_3 - 1)} + \frac{E(Q, p_3)}{\beta_3}$$

holds if $(Q, p_3, \beta_3) = (6, p_3, 1)$ for some prime $p_3 \geq 5$. That is $k = 2 \times 3 \times p_3$. Again, setting $\tau(k!) = 2\tau((k-1)!)$ yields $2(\alpha_1 + \alpha_2 + \alpha_3 + 3) = \alpha_1\alpha_2\alpha_3$. From observation (i), we know $\alpha_1 = E(6p_3, 2) > 3p_3 \geq 15$, $\alpha_2 = E(6p_3, 3) > 2p_3 \geq 10$ and $\alpha_3 = E(6p_3, p_3) \geq 6$. It is easy to verify that under these conditions $\alpha_1\alpha_2\alpha_3 > 2(\alpha_1 + \alpha_2 + \alpha_3 + 3)$.

Case (IV). k has 4 or more distinct prime factors.

We have $j \geq 4$ and

$$Q = \prod_{i=1}^{j-1} p_i^{\beta_i} \geq 2 \times 3 \times 5 = 30.$$

Since $2j + 1 < Q$, we have

$$2j + 1 < Q \frac{p_j^{\beta_j} - 1}{\beta_j(p_j - 1)} + \frac{E(Q, p_j)}{\beta_j}.$$

We have just shown $\tau(k!) \neq 2\tau((k-1)!)$ for the special cases when $\beta_i > \alpha_i/2j$. Now let's show $\tau(k!) < \sqrt{e}\,\tau((k-1)!) < 2\tau((k-1)!)$ for any other composite number $k \neq 4$ if $\beta_i \leq \alpha_i/2j$. Here $e = 2.71828\ldots$ is the Euler number.

$\beta_i \leq \alpha_i/2j$ implies $\beta_i < (\alpha_i+1)/2j$. We also know

$$\tau(k!) = (\alpha_1+\beta_1+1)(\alpha_2+\beta_2+1)\cdots(\alpha_j+\beta_j+1)(\alpha_{j+1}+1)\cdots(\alpha_t+1)$$

and

$$\tau((k-1)!) = (\alpha_1+1)(\alpha_2+1)\cdots(\alpha_j+1)(\alpha_{j+1}+1)\cdots(\alpha_t+1).$$

Since all factors after the $(j+1)$'st term are the same, it suffices if we just look at $(\alpha_1+\beta_1+1)(\alpha_2+\beta_2+1)\cdots(\alpha_j+\beta_j+1)$ and $(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_j+1)$. Subsequently, we have

$$\begin{aligned}(\alpha_1+1+\beta_1)\cdots(\alpha_j+1+\beta_j) &< \left(\alpha_1+1+\frac{\alpha_1+1}{2j}\right)\cdots\left(\alpha_j+1+\frac{\alpha_j+1}{2j}\right)\\ &= (\alpha_1+1)\cdots(\alpha_j+1)\left(1+\frac{1}{2j}\right)^j.\end{aligned}$$

Since $\lim\left(1+\frac{1}{2j}\right)^j = \sqrt{e}$ and it is a strictly increasing function, we have

$$\begin{aligned}(\alpha_1+1+\beta_1)(\alpha_2+1+\beta_2)\cdots(\alpha_j+1+\beta_j) &< \sqrt{e}(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_j+1)\\ &< 2(\alpha_1+1)(\alpha_2+1)\cdots(\alpha_j+1).\end{aligned}$$

In other words, $\tau(k!) < \sqrt{e}\tau((k-1)!) < 2\tau((k-1)!)$. □

Combining corollary 1 and theorem 3, we have:

THEOREM 7. *There are $\tau(k!)/2$ solutions to $\mathrm{S}(x) = k$ if and only if k is prime or $k = 4$.*

COROLLARY 8. *$\tau(k!) = 2\tau((k-1)!)$ if and only if k is prime or $k = 4$.*

Proof. This follows from theorem 4 if we let $\tau(k!)/2 = \tau(k!) - \tau((k-1)!)$. □

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Rex H. Wu is a physician at NYU Downtown Hospital who saw the airplane crashing into the World Trade Center on September 11, 2001. He couldn't imagine people would do such vicious acts. Only blocks away, he and his colleagues treated a few hundred victims at NYU Downtown Hospital that day. He also volunteered on-site the next couple days. He wishes to express his deepest sorrow to all the innocent lives lost during the attack. And his greatest respect goes to all the heroes on ground zero.