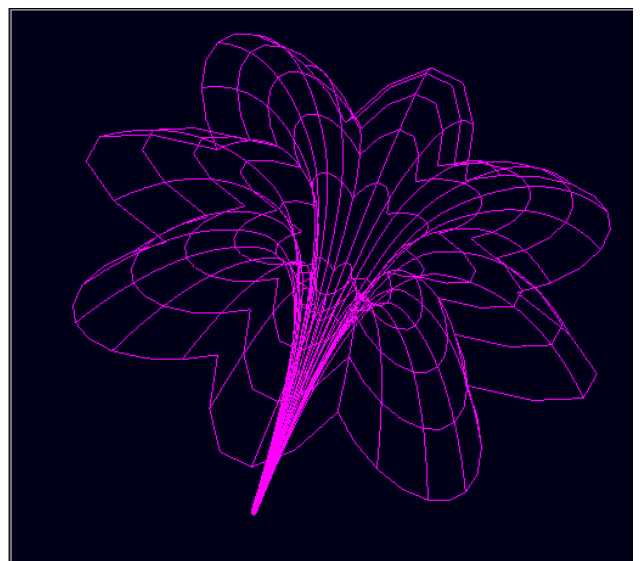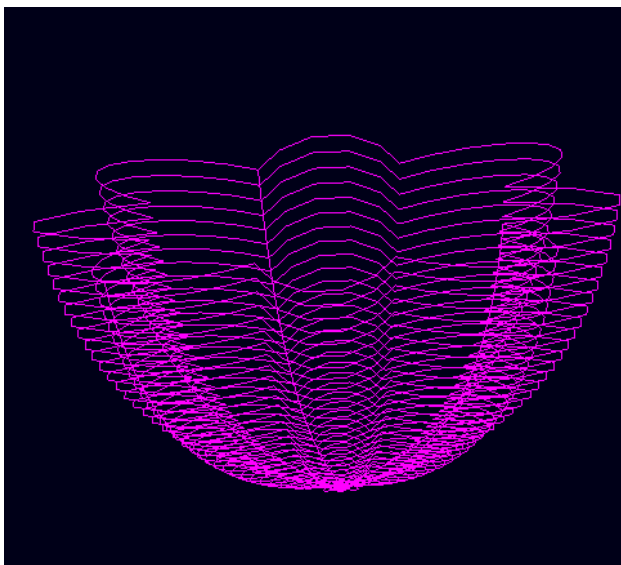# Modeling with Tensor Product B-Splines: Flower Shaped Objects With Rational Surfaces

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1. **INTRODUCTION**

Shape modeling with b-Splines has been a very popular and powerful technique in Computer aided geometric design. This can be attributed to the versatile nature of the b-spline representation which has some very desirable attributes like the convex hull properly and local control. Designing real world objects by specifying control points is an intuitive method and an efficient alternative to drawing objects by hand and hence b-splines have long been used in the design and animation industry.

2. **APPROACH**

This project aims at creating flower shaped objects using tensor product rational b-splines. This is accomplished by first creating a two dimensional b-spline curve based on the object to be created by specifying the control points interactively and making a surface by sweeping the curve along a two dimensional curve (path) that is in a plane perpendicular to the specified curve and that resembles the top view of a flower. This curve is rational and is made by a combination of elliptical arcs. The curve configuration uses two kinds of elliptical arcs and has totally 12 arcs. The angle subtended by these arcs at the center of the ellipse and the position of these arcs are fixed. However the lengths of the major axis and the minor axis of the ellipses from which these arcs are cut can be specified by the user.

3. **SPECIFYING THE CURVE TO BE SWEPT**

The user can make the curve by specifying the control points of the curve interactively. An initial knot vector with open end conditions is automatically created and this can be changed to a uniform floating knot vector to start with or can be further edited to get the desired knot vector.
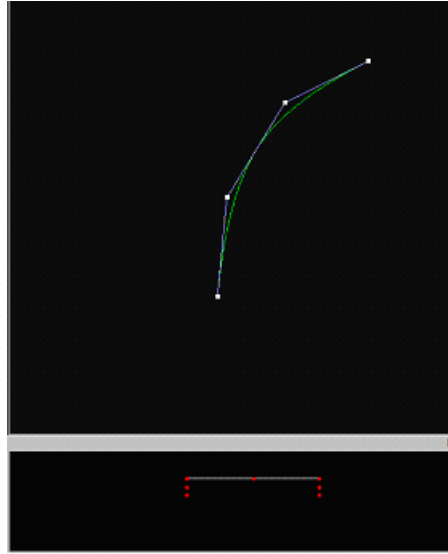
Fig 1

4. CONSTRUCTING THE FLOWER SHAPED CURVE

4.a BACKGROUND- ABOUT CONICS

Any quadratic can be written as

$$Ax^2+Bxy+Cy^2+Dx+Ey+F=0 \qquad\qquad (1)$$

This has six unknowns. However, if we fix the constant term to 1 say, by dividing the entire equation by the constant term, we have 5 unknowns. So any conic can be uniquely specified by five points (fig1).

[ref] shows how this problem can be reformulated and that any conic can be expressed as

$$L_1L_2xy+cL_3{}^2=0 \qquad\qquad (2)$$

where L_1 and L_2 (two tangents to the conic) and a line L_3 which passes through the points at which L_1 and L_2 are tangents- P_1 and P_3(Fig 2)are given. (This is equivalent to the previous formulation because we still have two points on the curve, two tangents and an unknown c (which accounts for the fifth degree of freedom).
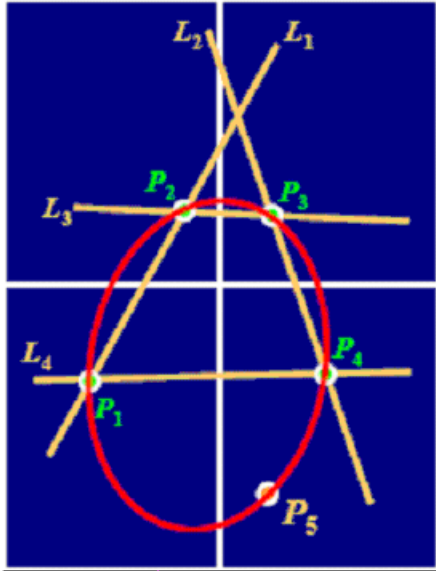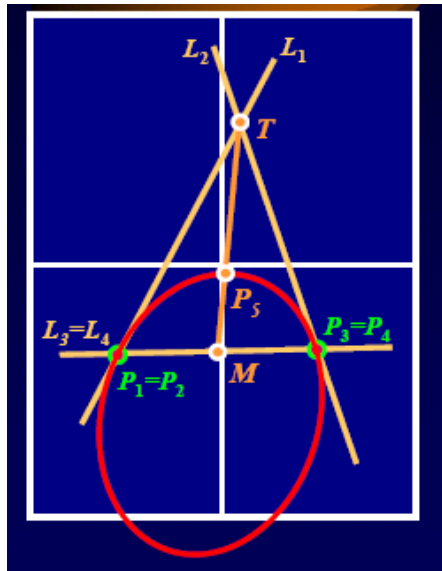
$$\rho = \|P_5 - M\| / \|T - M\|$$

Fig: 2a   Fig: 2b

[ref] further shows that if T is the intersection of L_1 and L_2 , M is the midpoint of the line segment P_1 and P_3 and P_5 the intersection of TM with the conic, the ratio ρ is a scalar which can uniquely specify the fifth point and hence the nature of the conic. If $\rho < \frac{1}{2}$ the resulting conic is an ellipse, if $\rho > \frac{1}{2}$, the resulting conic is a hyperbola and if $\rho = \frac{1}{2}$ the resulting conic is a parabola.

Now, suppose we have a pencil of conics

$$(1-\lambda)\, L_1 L_2 - \lambda L_3{}^2 = 0 \tag{3}$$

This is the same as equation 2, Where L_1, L_2 and L_3 have the same meaning as that in the previous paragraph, and a conic can be uniquely determined by specifying λ

Let K be a constant such that $\quad k = \dfrac{4\lambda}{1-\lambda}$ (4)

The relation between ρ and K (and hence λ) has been shown in the appendix and it is as follows

$$\rho = \frac{1}{1+\sqrt{\kappa}} \tag{5}$$

4.b CONICS AS RATIONAL BEZIER CURVES

[ref] shows that any conic section can be written as a quadratic rational parametric equation. The following is an expression for a conic section

$$\gamma(t) = \frac{w_1(b-t)^2 P_1 + 2w(t-a)(b-t)T + w_2(t-a)^2 P_3}{w_1(b-t)^2 + 2w(t-a)(b-t) + w_2(t-a)^2} \qquad (6)$$

as a parameter of t in the interval [a,b] where

$$\frac{w_1 w_2}{w^2} = \frac{4\lambda}{1-\lambda} = k \qquad (7)$$

Consider an ellipse with major axis 2a and minor axis 2b and with center at the origin. Its equation is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Let P_1 T P_3 be an arc of the ellipse as shown in the figure, where P_1 is an arbitrary point on the ellipse in the first quadrant. We can choose this point by using the parametric form of the ellipse and for some θ choosing (a cosθ, b sinθ) such that both coordinates of the point are positive. Now we choose the point P_3 by reflecting P_1 about the X axis. In this particular program, θ has been chosen as 60 degrees. So we have $P_1 = (\frac{a}{2}, \frac{b\sqrt{3}}{2})$ and

$$P_3 = (\frac{a}{2}, \frac{-b\sqrt{3}}{2}) \ .$$

Now T is computed by first computing the slope of the tangent by the following equation $[\frac{d}{dx}(f(x, y))] | P_1 = 0$. In our case, $\frac{dy}{dx} | P_1 = \frac{-b}{a\sqrt{3}}$ has been obtained.

Using the point slope form the equation of P_1T can now be computed.

Similarly, $\frac{dy}{dx} | P_2 = \frac{b}{a\sqrt{3}}$ and the equation of P_3T can also be computed .

Now the point of intersection T can be computed (by solving these equations) to be T=(2a,0) . Also here, M= (a/2,0) and P_5 is (a,0).

Therefore ρ can be calculated as 1/3 and hence K as 4 from the relationship between ρ and K that has been derived. (equation 1)
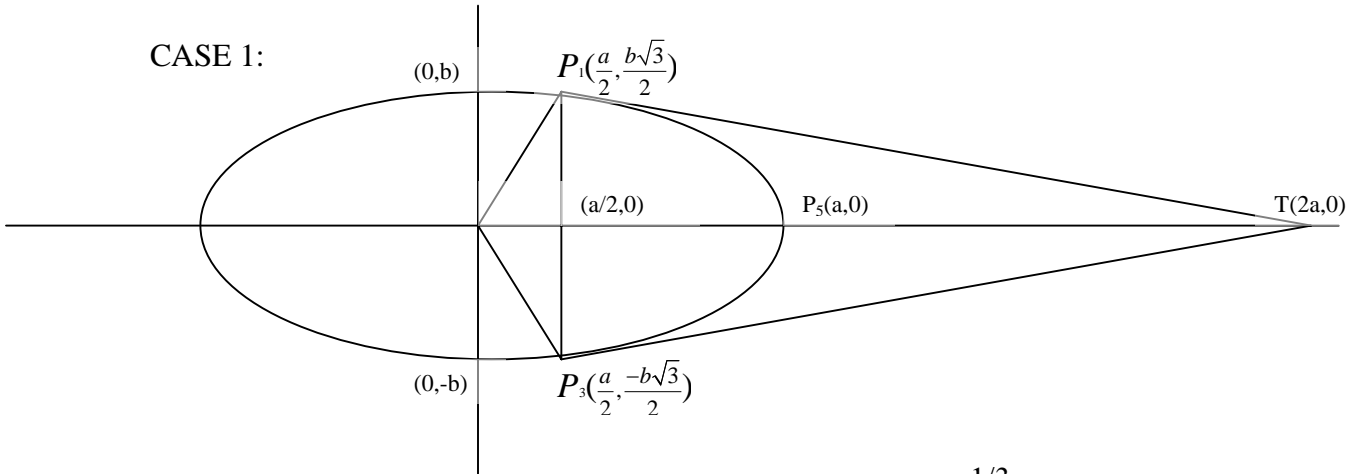
ELLIPSE AS A FRAGMENT OF THE FLOWER SHAPED CURVE

CASE 1:

Fig: 3a

$$\rho = 1/3$$

$$\kappa = \left(\frac{\rho - 1}{\rho}\right)^2 = 4$$

$$w_1 = 1, w_2 = 1 \quad \text{and} \quad w = \frac{1}{\sqrt{k}} = \frac{1}{2}$$
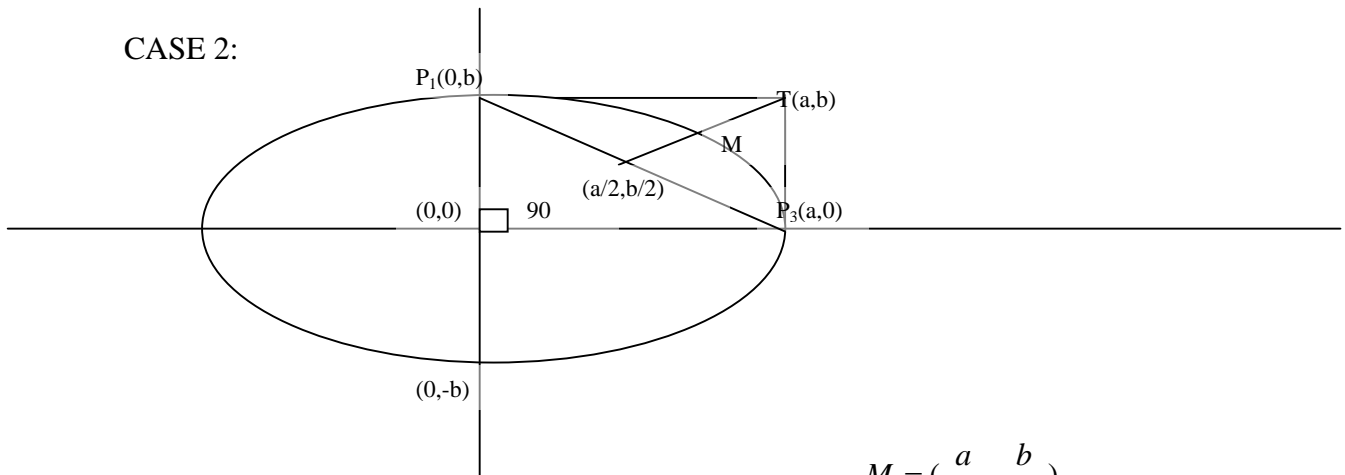
CASE 2:

Fig: 3b

$$M = \left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$$

$$\rho = \sqrt{2} - 1$$

$$\kappa = \left(\frac{\rho - 1}{\rho}\right)^2 = 2$$

$$w_1 = 1, w_2 = 1 \quad \text{and} \quad w = \frac{1}{\sqrt{k}} = \frac{1}{\sqrt{2}}$$

Similar calculations have been done to the second type of arc which subtends a right angle at the center. And the values obtained for ρ and K (and the coordinates of various intermediate points needed in the process) are shown in the figure.

We know that in our rational representation of the curve (equation 6), we know that

$$\frac{w_1 w_2}{w^2} = \frac{4\lambda}{1-\lambda} = k$$. Since we have the value of K, we choose an appropriate value

for weights w_1, w_2 and w such that the above condition (equation 7) is satisfied. In this case the values picked are shown in the figure 2(we have always picked w_1 and w_3 as one and picked w appropriately).

4c. COMBINING THE CONIC ARCS (BEZIER PIECES) TO FORM A B-SPLINE CURVE

To make a b-spline curve of these arcs, we need a knot vector and the control points.

Control Points

Since we want a closed curve, our first and last control point coincide. Therefore P_0 is equal to P_{24}. The rest of the points can be obtained by traversing each of our arcs in the proper order as shown in figure 4.

The semi major axis and the semi minor axis of the arc P_0, P_1 and P_2 are the values of a and b specified by the user respectively. The semi major axis and the semi minor axis of the arcs P_6, P_7 and P_8 , p_{12}, p_{13} and p_{14} and p_{18}, p_{19} and p_{20} are also obtained from a and b since these are reflected versions of the arc P_0, P_1 and P_2. Each diagonal petal has two arcs, each subtending 90 degrees at the center of its ellipse. The minor axis of the ellipse of the diagonal petals is basically the distance between P_0 and P_{20} and is therefore not specified by the user. The semi major axis of the arcs corresponding to the diagonal petals is the C value specified by the user.

Also in the above discussion, we have assumed that each ellipse is centered at the origin. Therefore the control points are translated to the desired position when needed.

Knot Vector

Since we want to have open end conditions, the first three and the last three knots are the same. Since we want to interpolate every alternate point in the control polygon, we repeat internal knots twice as shown in the figure 4.
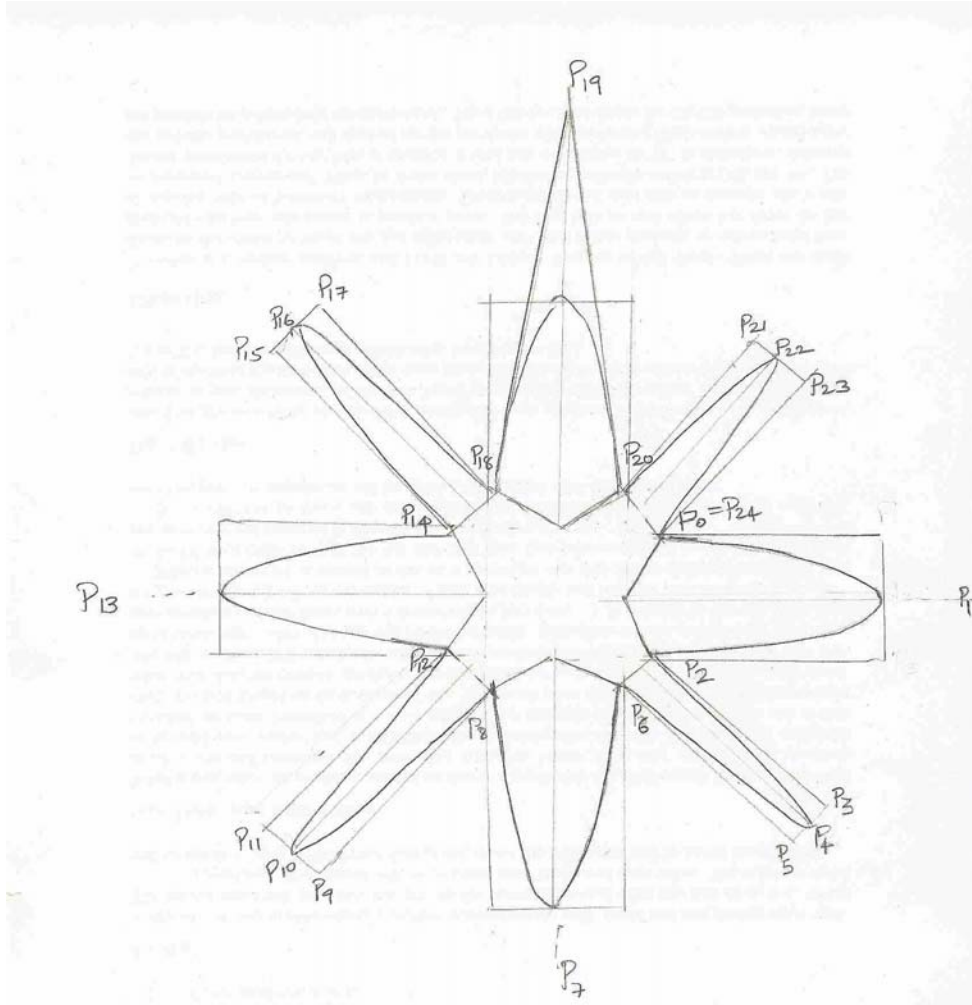
Weights:

$$1,\frac{1}{2},1,\frac{1}{\sqrt{2}},1,\frac{1}{\sqrt{2}},1,\frac{1}{2},1,\frac{1}{\sqrt{2}},1,\frac{1}{\sqrt{2}},1,\frac{1}{2},1,\frac{1}{\sqrt{2}},1,\frac{1}{\sqrt{2}},1,\frac{1}{2},1,\frac{1}{\sqrt{2}},1,\frac{1}{\sqrt{2}},1$$

Knot Vector:

0 0 0 1 1 2 2 3 3 4 4 5 5 6 6 7 7 8 8 9 9 10 10 11 11 12 12 12

Fig 4a: The figure above has a total of 12 elliptical arcs. The diagonal petals have two arcs each which subtend 90 degrees at the center of their ellipse and four arcs which make $Tan^{-1}(\sqrt{3}b/a)$ where a and b are the major and minor axes of the corresponding ellipses. The minor axis of the ellipse corresponding to the diagonal petals are fixed and their major axis can be specified by the user (the parameter C)
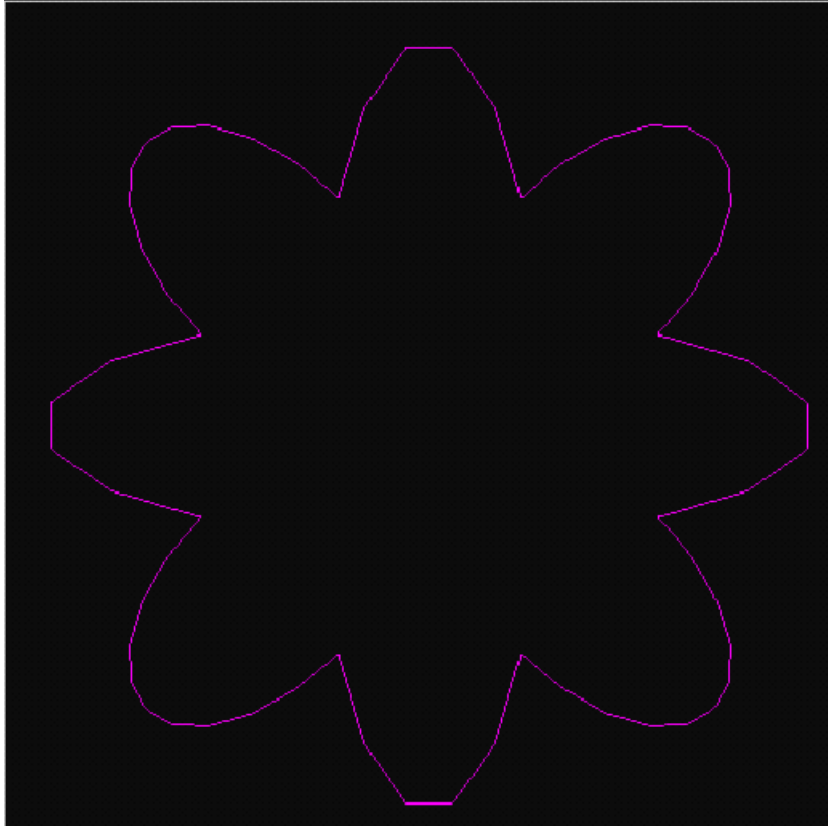
Fig 4b

5. MAKING THE TENSOR PRODUCT SURFACE

Definition[ref]: If F and G are two sets of univariate functions with interval domains U and

V respectively, $F = \{f_i(u)\}_{i=0}^{m}$ and $G = \{g_j(v)\}_{j=0}^{n}$ the surface, formed by

$$h(u,v) = \sum_{j=0}^{n}\sum_{i=0}^{m} c_{i,j}\, f_i(u)g_j(v)$$ is a tensor product surface with domain U x V. If $c_{i,j}$

belongs to R^3 for all i, j, then h is a parametric surface.

Suppose the collections of functions are $F = \{B_{i,\,k_u,\,\tau_u}(u)\}_{i=0}^{m}$ and

$G = \{B_{j,\,k_v,\,\tau_v}(v)\}_{j=0}^{n}$ where τ_u is a knot vector with $m+k_u+2$ non-decreasing elements

and τ_v is a knot vector with $n+k_v+2$ non-decreasing elements,

$$\sigma(u,v) = \sum_{j=0}^{n} \sum_{i=0}^{m} P_{i,j} B_{i,k_u,\tau_u}(u) B_{j,k_v,\tau_v}(v)$$

is a degree K_u x K_v tensor product B-spline surface over the rectangular domain

$$[\tau_{u,k_u}, \tau_{u,m+1})[\tau_{v,k_v}, \tau_{v,n+1}) \subset \Re^2$$

We have a knot vector in the u and the v direction and we need to construct a set of control points belonging to $\Re^3$ from the control points of the two individual curves. Let the control points of the two individual curves $\gamma(u)$: the flower shaped curve in the xy plane and $\alpha(v)$: the arbitrary curve in the xz plane belonging to $\Re^2$ be $C_\gamma = \{A_{\gamma i}(x_j, 0, z_j, 1)\}_{i=0}^{n}$ and

$C_\alpha = \{A_{\alpha i}\}_{i=0}^{m}$. The control points of the surface $\sigma(u,v)$ we want are constructed as follows.

$$C_\sigma = \{A_{\sigma ij}\} \forall\ i = 0\ to\ n\ and\ j = 0\ to\ m \quad \text{and} \quad A_{\sigma ij} = x_j A_{\alpha i} + (0, 0, z_j, 0)$$

Hence $\sigma(u,v) = \sum_{i,j} A_{\sigma ij} B_{i,2,u}(u) B_{j,k,v}(v)$

Fig 5a

Fig 5b

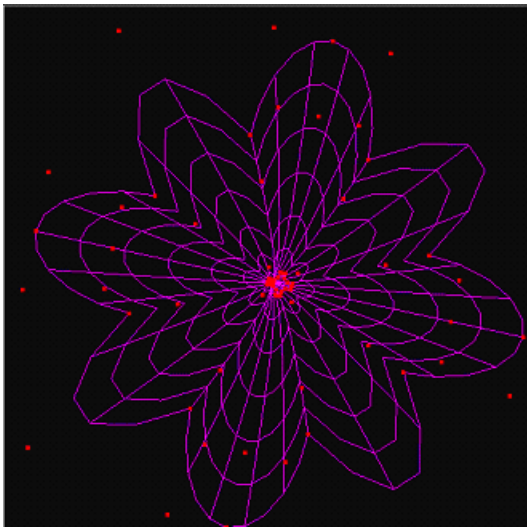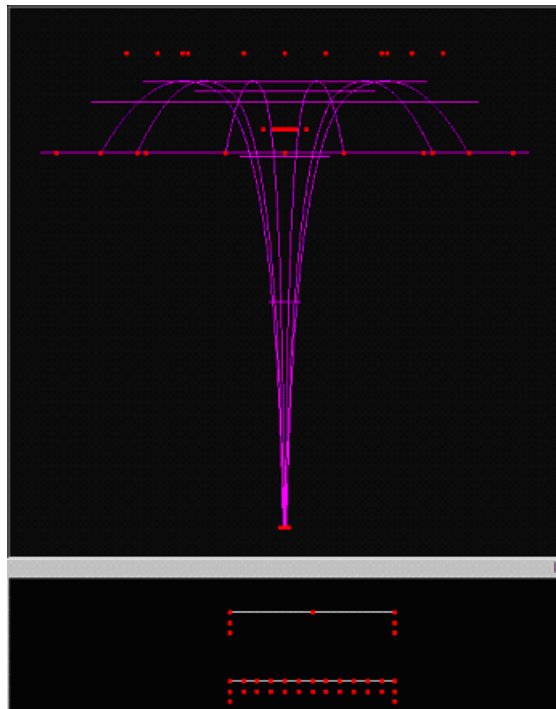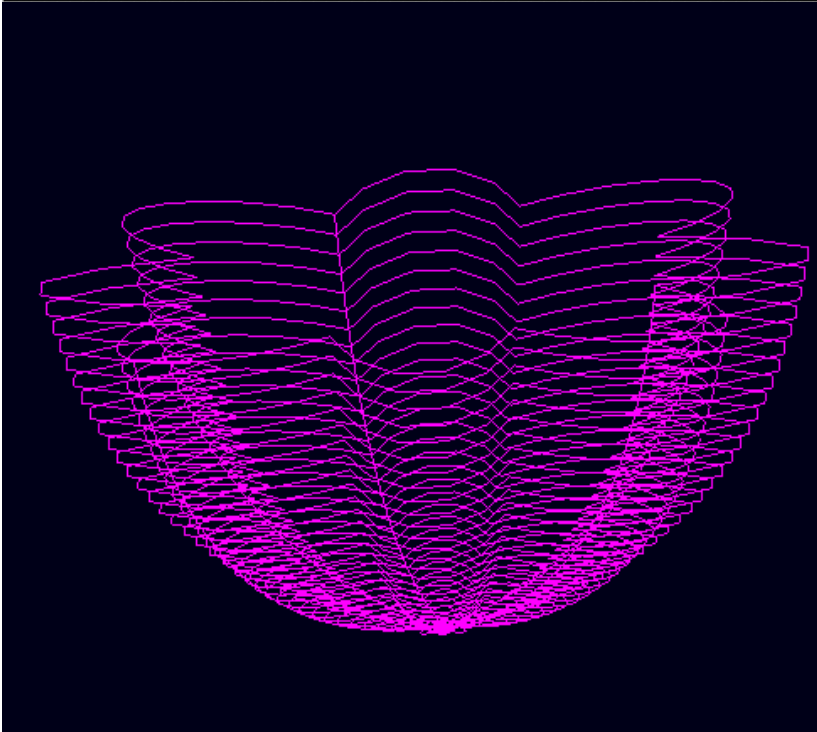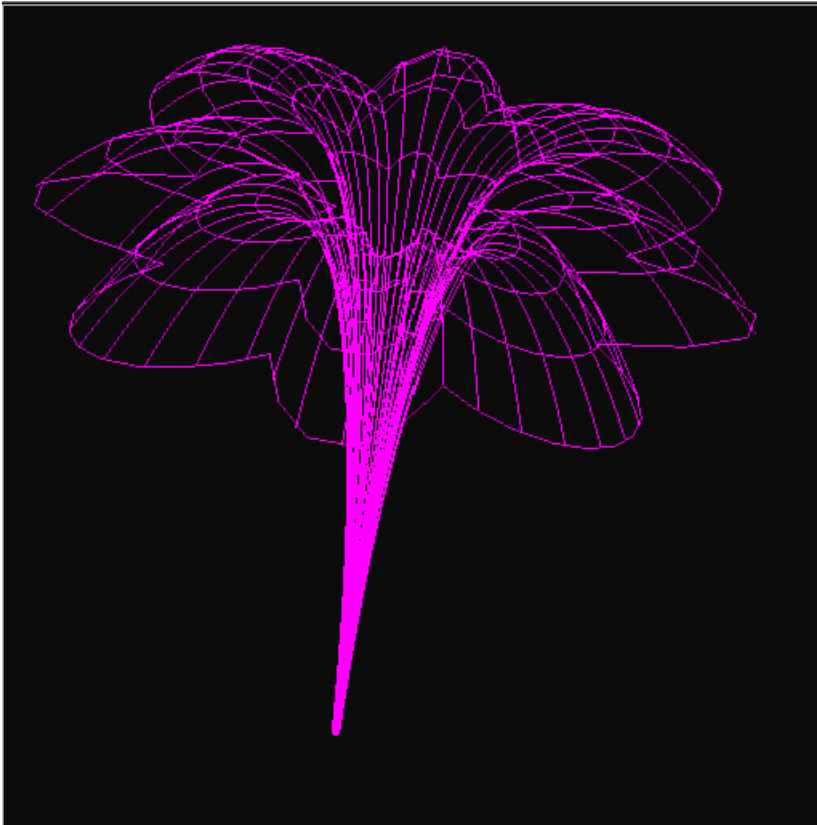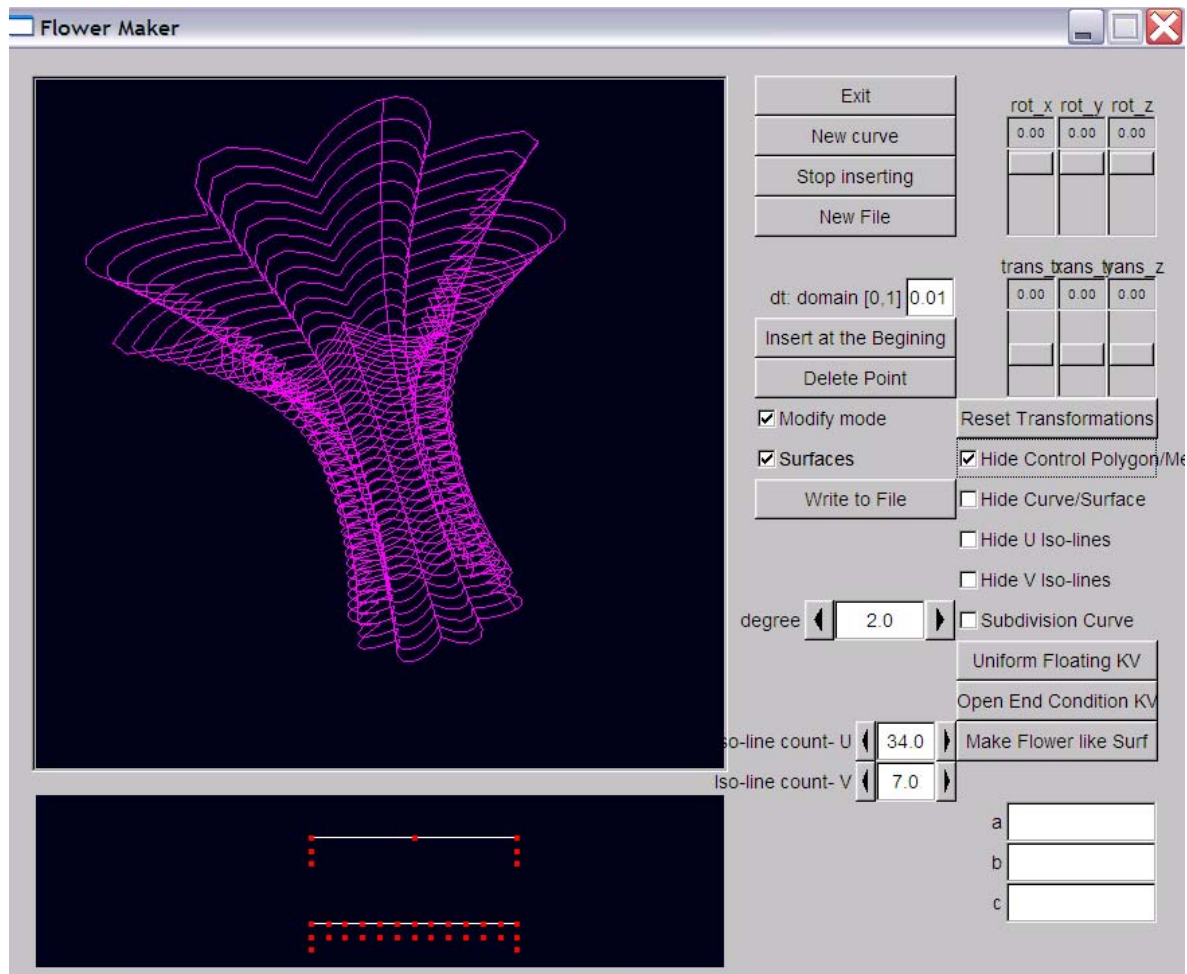## 6. SOME RESULTS

7. THE GUI:

To use the GUI to make a flower shaped surface first click on the "Make Surface" button. Then click on the "New Curve" button and make the curve by clicking on the screen to specify the control points (Use the GUI to modify the curve if needed). And click on "make flower shaped surface" to make the surface. The object can be translated and rotated either using the GUI or the mouse. (Left mouse for rotation, right mouse for translation and center for zooming)

The GUI contains buttons, text boxes, menus and sliders. A brief description of each item in the GUI is given below.

1. EXIT: Quit the program

2. NEW_CURVE: Clears the contents of the screen. New control points can be inserted by clicking on the screen. Every click positions a new control point and this continues until we press the stop inserting button.

3. STOP INSERTING: Stops inserting control points on mouse clicks. We can however use normal selection and editing/inserting to modify/insert points.

4 NEW FILE: Opens the specified file and displays the curve.

5. dt: Changes the dt value (increment of parameter) for doing the constructive algorithm. So if we change this we can see a change in the resolution (number of samples plotted).

6 INSERT AT THE BEGINNING: When we are in the insert mode (item 8) we can select a point and insert a point after it. When we click this button (insert at the beginning) the position where we click on the screen is made the first control point.

7. DELETE POINT: If we select a point. Clicking on this deletes the selected point.

8. MODIFY POINT: If this is checked, we are in modify mode. i.e., if we select a point and click elsewhere, the location of the point is modified. If this is unchecked, a new control point is inserted.

9: SURFACE: Specify if we are in surface mode or curve mode. If we are in curve mode, the initial curve which we interactively specified is displayed.

10. WRITE TO FILE :Write the current mesh base/refined to file.

11. DEGREE: Degree of the curve being constructed

12. U-ISO-LINE-COUNT: Number of lines to be drawn in u direction

13. V-ISO-LINE-COUNT: Number of lines to be drawn in v direction

14. ROT X, ROT Y, ROT Z: Rotate the object using GUI

15. TRANS X, TRANS Y, TRANS Z: Translate the object using GUI

16. RESET TRANSFORMATION: reset all the rotation and translation transformations

17. HIDE CONTROL POLYGON: hide the control polygon

18. HIDE SURFACE : do not show the surface

19. HIDE U ISO LINES: Not to show iso lines in the V direction

20. HIDE V ISO LINES: Not to show iso lines in the V direction

21. SUBDIVISION CURVE: When displaying the curve (in curve mode) use the control polygon of the refined curve to display the curve

22. UNIFORM FLOATING KNOT VECTOR: Make uniform floating knot vector (When creating curves in the curve mode)

23. OPEN END CONDITION KNOT VECTOR: Make an open end condition knot vector (When creating curves in the curve mode)

24. MAKE FLOWER LIKE SURFACE: Make the flower like surface from the curve specified.

25. NEW SURFACE: Start creating a new surface.

26. a b and c: a and b specify the a and b parameters for creating the flower shaped curve (and and b are the semi major and semi minor axis of the ellipse that subtends an angle $\mathrm{Tan}^{-1}(\sqrt{3}b/a)$ at the center and c specifies the major axis of the arc that subtends 90 degrees at the center of its ellipse .The minor axis of this ellipse (corresponding to the second type of curve) is automatically computed when a, b and c are specified).

EDITING THE CURVE

All the points can be changed and new points can be added.

To modify a control point: Click on the point. The point is selected and it is highlighted by being drawn in a different color. Click on the new position of the point. This changes the curve

To insert a point: Click on the point next to which you want to insert this point. The point is selected and it is highlighted by being drawn in a different color. Click on the position where you want to insert it. The point is inserted

Note: During selection, if the point is not highlighted when you try to select it, please click on it again

Relation between λ, K and ρ

Consider the equation of the conic in the transformed coordinate system (the tangents given being the u and v axes and OP_0 and OP_2 being unity).

$$(1-\lambda)\, uv - \lambda\,(u+v-1)^2 = 0 \longrightarrow \text{①}$$
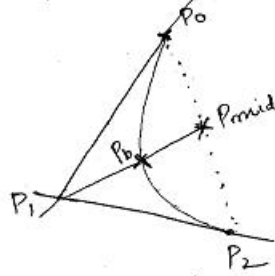
$$P_{mid} = \frac{P_0 + P_2}{2}$$

$$= \left(\frac{1}{2}, \frac{1}{2}\right)$$

$$|P_1 P_{mid}| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

$$= \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

Equation of $P_1 P_{mid}$ is $u - v = 0$.

Using this in ① and eliminating v, we get-

$$(1-\lambda)\, u^2 - \lambda\,(2u-1)^2 = 0$$

$$\Rightarrow \frac{\lambda}{1-\lambda} = \frac{u^2}{(2u-1)^2} \Rightarrow \left(\frac{2u}{2u-1}\right)^2 = \frac{4\lambda}{1-\lambda} = K.$$

$$\frac{2u}{2u-1} = \pm\sqrt{K} \Rightarrow u = \frac{1}{2}\,\frac{\sqrt{K}}{1\pm\sqrt{K}}.$$

We choose $\frac{1}{2}\,\frac{\sqrt{K}}{1+\sqrt{K}}$ since we want u always $< \frac{1}{2}$

(The curve intersects the line in 2 points. We want the point inside the triangle).

Now $\rho = \dfrac{|P_b\, P_{mid}|}{|P_1\, P_{mid}|}$

$$|P_b\, P_{mid}| = \sqrt{\left(\frac{1}{2} - \frac{1}{2}\frac{\sqrt{K}}{1+\sqrt{K}}\right)^2 + \left(\frac{1}{2} - \frac{1}{2}\frac{\sqrt{K}}{1+\sqrt{K}}\right)^2}$$

$$= \sqrt{\frac{2}{4}\left(1 - \frac{\sqrt{K}}{1+\sqrt{K}}\right)}$$

$$= \frac{1}{\sqrt{2}}\,\frac{1+\sqrt{K}-\sqrt{K}}{1+\sqrt{K}} = \frac{1}{\sqrt{2}}\,\frac{1}{1+\sqrt{K}}$$

$$\rho = \frac{\frac{1}{\sqrt{2}}\,\frac{1}{1+\sqrt{K}}}{\frac{1}{\sqrt{2}}} = \frac{1}{1+\sqrt{K}}$$

$$\Rightarrow \boxed{\rho = \frac{1}{1+\sqrt{K}}}$$