A Brief on Asymptotics

(Preliminary Draft Version: January 11, 2003)

L.P. Lebedev
Department of Mechanics and Mathematics
Rostov State University, Russia
&
Department of Mathematics
National University of Colombia, Colombia

and

M.J. Cloud
Department of Electrical and Computer Engineering
Lawrence Technological University, USA

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1 Introduction

All engineering students have certainly made use of the Taylor expansions for sinusoids, exponentials, etc. They probably also remember something about the corresponding remainder terms, and the rest of the associated (and difficult) material presented by their former calculus professors. Many students know about function analyticity and, moreover, are aware that not all functions are analytic. However few students have thought about the difference between the

Taylor expansion of $\sin x$, say, and the Taylor expansion of some function near a point at which the function is not holomorphic (i.e., the Taylor series does not converge in any neighborhood of the point). There are functions for which one can compose the Taylor series at some point x_0, but for which this series fails to converge at any point near x_0. It may even be that only finitely many of the Taylor coefficients of a function exist, making it impossible to compose the whole Taylor series. However, the reader understands that our main practical goal in deriving the Taylor series of a function is the ability to calculate its values or, what amounts to the same thing, a knowledge of its behavior in some neighborhood of a point using simple formulas. For this the Taylor expansion having the form

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + o\left(|x - x_0|^n\right) \quad (1)$$

is quite appropriate, even if we know nothing about the $n + 1$-th derivative. Indeed we know that $f(x)$ is approximated by the expression

$$f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \quad (2)$$

in a neighborhood of x_0 within accuracy $o\left(|x - x_0|^n\right)$.[1] Thus if $|x - x_0|$ is sufficiently small we know that the error produced in using formula (2) instead of $f(x)$ is of the order of $|x - x_0|^n$. Of course, it would be better to have a more exact estimation of the error, and for that in calculus there were derived various forms of the remainder term that allow us to evaluate a bound for the error. Thus expansion (1) provides us with a way to calculate an approximation to a function in some neighborhood using comparatively simple formulas. In this way Napier composed the first table of logarithms. Many tables of values for special functions could be calculated at a time when pencil and paper were the only available tools.

Taylor expansions of functions are the first and foremost examples of *asymptotic expansions* (or "asymptotics," for short). The main feature of asymptotic expansions is that they approximate the behavior of functions or other expressions using comparatively simple expressions and propose some bounds for the approximation errors. Sometimes an asymptotic, a simple analytic expression, is all we can obtain to characterize the behavior of a solution to a physical problem at a singular point. In mechanics this could be a boundary point at

[1] We recall that the notation $f(x) = o(g(x))$ when $x = x_0$ means

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0.$$

We will use the notation $f(x) = O(g(x))$ when $x = x_0$ if there is a neighborhood of x_0 and a constant C such that in this neighborhood the relation

$$\left| \frac{f(x)}{g(x)} \right| \le C$$

holds.

which the type of clamping of a body changes, or a point of jump discontinuity in the external forces acting on a body, or a corner point of a body at which the behavior of the solution differs from its behavior at all other points of the body. Quite frequently a knowledge of this singular asymptotic approximation permits us to compose a simple and fast algorithm to find a numerical solution of the problem, whereas direct algorithms that do not take into account the nature of the solution at a singular point can give incorrect results. Asymptotic expansions are widely used in physics and other mathematical sciences for finding the dependence on a parameter of integrals, solutions of boundary value problems for ordinary and partial differential equations, eigenfrequencies of various systems, etc. Asymptotics from various fields of application all share a common basic nature, but the methods needed for their construction can differ widely. Some of these methods require only a knowledge of calculus, but more complex problems require more complex tools to construct the asymptotics. Having in mind the structure of the Taylor expansion, let us extend the idea of asymptotic expansions.

2 The main definitions

Let us note first that a simple change of the variable $x - x_0$ to x allows us to consider asymptotics at $x = 0$. Next, we note that the powers x^n are not the only functions with respect to which we can compose asymptotic expansions. For example, consider $y = \sqrt{x}\cos x$ at $x = 0$. There is no Taylor expansion of this function (except the trivial one $y = \sqrt{x}\cos x = o(|x|)$) at $x = 0$. However, it is clear that the expansion

$$\sqrt{x}\cos x = \sqrt{x} + \sum_{n=1}^{\infty}(-1)^n \frac{x^{2n}\sqrt{x}}{(2n)!} \tag{3}$$

plays the same role as the Taylor expansion. The reader could raise the objection that this is composed of the Taylor expansion; however, he should envision a situation in which we lack any knowledge of the explicit form of the function but have obtained the right-hand side of (3) as a solution to some boundary value problem which we could not solve exactly. Thus we have an expansion in $x^n\sqrt{x}$ that makes sense only for nonnegative x and for small x; the inclusion of a few terms of the sum presents us with an accurate approximation to the solution. Let us summarize what we expect to have when introducing a general form of asymptotic expansion:

1. The generalization can use not only powers x^n as base functions.

2. An asymptotic expansion is composed as a sum of sufficiently simple expressions.

3. The addition of a term to the asymptotic expression brings a higher accuracy of approximation.

4. The Fourier transform and some other integral transformations introduce complex parameters into consideration. The theory should cover the case of a complex variable.

Having this in mind, let us introduce the main elements of asymptotics.

Asymptotic expansion

Suppose $f = f(z)$ depends on a complex variable z. We suppose that z_0 is a limit point of the domain D where $f = f(z)$ is determined and where an asymptotic expansion will be constructed. Suppose we have a sequence of functions $\{\varphi_n(z)\}$, $n = 1, 2, 3, \ldots$, each of which is determined on the intersection of D and some neighborhood of z_0. This sequence is called asymptotic if for each n we have

$$\varphi_{n+1}(z) = o(\varphi_n(z)) \quad \text{when } z \to z_0. \tag{4}$$

Note that here we require z to tend to z_0 being in D.

Definition 2.1. Let $\{\varphi_n(z)\}$, $n = 1, 2, 3, \ldots$, be an asymptotic sequence. A formal series $\sum_{n=1}^{\infty} a_n \varphi_n(z)$ is called an *asymptotic expansion* (or an *asymptotic*) of $f = f(z)$ at z_0 if for each $k \geq 1$ we have

$$\left| f(z) - \sum_{n=1}^{k} a_n \varphi_n(z) \right| = o(\varphi_k(z)) \quad \text{when } z \to z_0,\ z \in D. \tag{5}$$

We will denote the fact that $\sum_{n=1}^{\infty} a_n \varphi_n(z)$ is an asymptotic expansion of $f(z)$ as

$$f(z) \sim \sum_{k=1}^{\infty} a_k \varphi_k(z)$$

The special notation "$\sim$" is used to remind us that an asymptotic expansion is a *formal series*: convergence of the series is neither implied nor precluded here.

Example 2.1. For a power expansion at zero we would write

$$f(z) \sim a_0 + \sum_{k=1}^{\infty} a_k z^k.$$

Remark 2.1. We shall reserve the same term "asymptotic" for a finite sum $\sum_{n=1}^{N} a_n \varphi_n(z)$ if (5) is fulfilled for each $k \leq N$.

Remark 2.2. We leave it to the reader to demonstrate that (5) in Definition 2.1 can be written in the equivalent form

$$\left| f(z) - \sum_{n=1}^{N} a_n \varphi_n(z) \right| = O(\varphi_{N+1}(z)) \quad \text{when } z \to z_0,\ z \in D. \tag{6}$$

So the *truncation error* must be of the order of the first term omitted. We shall use (5) or (6) interchangeably.[2]

[2]However, they are not quite equivalent in the case of finite expansion to N terms only.

The expression $a_1\varphi_1(z)$ is called the *main term* of the asymptotic. There are times in which we can find only this term, but are nonetheless pleased that we can do so.

It is evident that $1, x, x^2, \dots, x^n, \dots$ is an asymptotic sequence at $x = 0$ for a function of a real variable. Similarly, in the complex plane the sequence $1, z, z^2, \dots, z^n, \dots$ is asymptotic at $z = 0$.

Exercise 2.1. Which of the following are asymptotic sequences at the indicated points:

(a) $1, x, x^2, \dots, x^n, \dots$ at $x = 1$;

(b) $1, x^2, x^4, \dots, x^{2n}, \dots$ at $x = 0$;

(c) $1, \sin x, \sin 2x, \sin 3x, \dots, \sin nx, \dots$ at $x = 0$.

We have said that the Taylor expansion is a particular case of an asymptotic expansion. The reader should verify that the Taylor expansion fits the definition of the asymptotic expansion.

The Laurent series at a pole of a complex function is another example of an asymptotic expansion.

Similarly to the case of asymptotics for finite z_0 we can consider asymptotics of functions at infinity. Even on the real axis the behavior of a function can be different depending on whether $x \to +\infty$ or $x \to -\infty$. Hence we should expect to find that the neighborhood of infinity, i.e., the set $|z| > N$, is normally partitioned into some parts where the asymptotic behavior of a function is different. A function analytic at infinity has an expansion

$$f(z) = b_0 + \sum_{n=1}^{\infty} \frac{b_n}{z^n}.$$

This presents an example of an asymptotic expansion of a function at infinity, and $1, z^{-1}, z^{-2}, z^{-3}, \dots$ is an example of an asymptotic sequence at an infinite point. Later we shall see asymptotic sequences that do not belong to the class of power sequences.

Simple properties of asymptotic expansion

It is easy to see that the coefficients a_n of an asymptotic expansion of $f(z)$ are defined uniquely by

$$a_n = \lim_{z \to z_0} \frac{f(z) - \sum_{k=1}^{n-1} a_k \varphi_k(z)}{\varphi_n(z)}. \tag{7}$$

Since an asymptotic expansion is a formal sum of terms, we expect to be able to apply simple arithmetic operations to such an expansion.

Let $\{\varphi_n(z)\}$ be an asymptotic sequence at $z = z_0$. Then the following properties hold, as the reader can verify by the use of Definition 2.1.

1. Let c be a constant. If $\sum_{n=1}^{\infty} a_n \varphi_n(z)$ is an asymptotic expansion of $f(z)$ at z_0, then $cf(z)$ has asymptotic expansion $\sum_{n=1}^{\infty} ca_n \varphi_n(z)$.

2. Let, in addition, $\sum_{n=1}^{\infty} b_n \varphi_n(z)$ be an asymptotic expansion for $g(z)$ at $z = z_0$. Then $f(z) + g(z)$ has asymptotic expansion $\sum_{n=1}^{\infty}(a_n + b_n)\varphi_n(z)$ at the same point in the common domain where both asymptotics are valid.

Properties of power-type asymptotic expansions. Asymptotic expansions for the product and quotient of two functions are difficult to formulate in terms of general asymptotic sequences. We have seen that the asymptotic power sequence is an important tool in applications (the reader should remember that for most points of solutions of boundary value problems, or of other functions that usually arise in applications, the Taylor expansion works perfectly; the points at which we need other types of asymptotic expansions are exceptional). Let us discuss other properties of asymptotic power-type expansions.

We have said that the change of variable $z - z_0$ to z moves an arbitrary finite point z_0 at which an asymptotic expansion is sought to $z_0 = 0$. Moreover, the change $1/z$ to z moves the infinitely distant point to zero as well. Thus considering the properties of a power-type asymptotic expansion of $f(z)$ at zero,

$$a_0 + \sum_{k=1}^{\infty} a_k z^k$$

we study simultaneously asymptotic expansions at z_0,

$$a_0 + \sum_{k=1}^{\infty} a_k (z - z_0)^k$$

and the asymptotics at infinity,

$$a_0 + \sum_{n=1}^{\infty} \frac{a_n}{z^n}.$$

Thus we suppose that $f(z)$ has an asymptotic expansion at $z = 0$, which means that for each n there holds

$$f(z) = a_0 + \sum_{k=1}^{n} a_k z^k + o(z^n) \quad \text{as } z \to 0. \tag{8}$$

The function $g(z)$ has a similar asymptotic expansion at zero:

$$g(z) = b_0 + \sum_{k=1}^{n} b_k z^k + o(z^n) \quad \text{as } z \to 0. \tag{9}$$

It is clear that the above two properties of asymptotics are valid in this case. Now we continue.

3. $f(z) \cdot g(z)$ has the asymptotic expansion

$$c_0 + \sum_{k=1}^{\infty} c_k z^k$$

where

$$\begin{aligned} c_0 &= a_0 b_0, \\ c_1 &= a_0 b_1 + a_1 b_0, \\ c_k &= a_0 b_k + a_1 b_{k-1} + \cdots + a_k b_0, \\ &\vdots \end{aligned}$$

This follows from direct multiplication of the left and right parts of (8) by the corresponding parts of (9). Note that for finite asymptotics (i.e., for the case in which (8) or (9) are valid only up to some n_1 or n_2, respectively) the product power asymptotic has order equal to the least of n_1 and n_2. The situation is quite similar to that in which we multiply approximations of decimal numbers: the product has the least number of valid digits possessed by the multipliers.

4. If $g(0) = b_0 \neq 0$ then $f(x)/g(x)$ has a power-type asymptotic expansion $d_0 + \sum_{k=1}^{\infty} d_k z^k$ at zero where the coefficients of the expansion can be found by formula (7):

$$\begin{aligned} d_0 &= \frac{a_0}{b_0}, \\ d_1 &= \lim_{z\to 0} \frac{\frac{f(z)}{g(z)} - \frac{a_0}{b_0}}{z} = \frac{a_1 b_0 - a_0 b_1}{b_0^2}, \\ d_2 &= \lim_{z\to 0} \frac{\frac{f(z)}{g(z)} - d_0 - d_1 z}{z^2} = \frac{a_2 b_0^2 - a_1 b_0 b_1 + a_0(b_1^2 - b_0 b_2)}{b_0^3}, \\ d_3 &= \lim_{z\to 0} \frac{\frac{f(z)}{g(z)} - d_0 - d_1 z - d_2 z^2}{z^3} \\ &= \frac{a_3 b_0^3 - a_2 b_0^2 b_1 + a_1(b_0 b_1^2 - b_0^2 b_2) - a_0(b_1^3 - 2b_0 b_1 b_2 + b_0^2 b_3)}{b_0^4}, \end{aligned}$$

etc.

Let us also consider the operations of differentiation and integration. We shall do this for a function of a real variable x.

5. Suppose $f(x)$ has an asymptotic expansion

$$f(x) \sim a_0 + \sum_{k=1}^{\infty} a_k x^k \tag{10}$$

at zero, and it is known that its derivative is continuous in some neighborhood of zero and has an asymptotic expansion at zero. Then the asymptotic expansion for $f'(x)$ at zero can be obtained by formal term-by-term differentiation of the expansion (10):

$$f'(x) \sim a_1 + \sum_{k=2}^{\infty} a_k x^{k-1}.$$

Note that we should suppose the existence of an asymptotic expansion for $f'(x)$. The sufficient conditions for this are given by various versions of calculus theorems on the Taylor expansion. So, let the expansion for $f'(x)$ be

$$f'(x) \sim c_0 + \sum_{k=1}^{\infty} c_k x^k.$$

The Newton–Leibnitz formula

$$f(x) - f(0) = \int_0^x f'(s)\, ds$$

and substitution of the finite asymptotic expansion for $f'(x)$ give us

$$\begin{aligned} f(x) - a_0 &= \int_0^x \left(c_0 + \sum_{k=1}^{n} c_k s^k + O(s^{n+1}) \right) ds \\ &= c_0 x + \sum_{k=1}^{n} c_k \frac{x^{k+1}}{k+1} + O(x^{n+2}) \qquad (11) \end{aligned}$$

since we can integrate a finite sum term-wise. Expression (11) means its right-hand side (together with a_0) is an asymptotic expansion of $f(x)$. Using uniqueness of the asymptotic expansion and identifying the terms in (11) with those in (10), we complete the proof.

6. Suppose that $f(x)$ is continuous in a neighborhood of $x = 0$ and has an asymptotic expansion (10). Then $\int_0^x f(s)\, ds$ has an asymptotic expansion at $x = 0$ that is derived by the term-wise integration

$$\int_0^x f(s)\, ds \sim \sum_{k=0}^{\infty} \frac{a_k x^{k+1}}{k+1}.$$

The proof follows from the chain of transformations

$$\int_0^x f(s)\, ds = \int_0^x \left(a_0 + \sum_{k=1}^{n} a_k s^k + O(s^{n+1}) \right) ds = \sum_{k=0}^{n} \frac{a_k x^{k+1}}{k+1} + O(x^{n+2}).$$

7. It is useful to know how to integrate a function having a power-type asymptotic expansion at an infinitely remote point:

$$f(x) \sim b_0 + \sum_{k=1}^{\infty} \frac{b^k}{x^k}.$$

Suppose that $f(x)$ is continuous when $x > M > 0$. Then the function

$$\Psi(x) = \int_x^{\infty} \left(f(s) - b_0 - \frac{b_1}{s} \right) ds \tag{12}$$

is continuous when $x > M$ and has an asymptotic expansion

$$\Psi(x) \sim \sum_{k=2}^{\infty} \frac{b^k}{(k-1)x^{k-1}}$$

at $x \to \infty$. For it is clear that the integrand in (12) is continuous in $[M, \infty)$ and is majorized by C/s^2. Thus the integral (12) converges and is a continuous function. Using the definition of asymptotic expansion for $f(x)$ we get

$$\Psi(x) = \int_x^{\infty} \left(\sum_{k=2}^{n} \frac{b^k}{s^k} + O\left(\frac{1}{s^{n+1}} \right) \right) ds.$$

Integrating term-wise we obtain

$$\Psi(x) = \sum_{k=2}^{n} \frac{b^k}{(k-1)x^{k-1}} + O\left(\frac{1}{x^n} \right) \quad \text{as } x \to \infty.$$

So the integration of an expansion at infinity can be performed term-wise. We had to remove the non-integrable terms from the function first, however.

It is well known how to find the simplest kinds of asymptotics: the Taylor expansions. Now let us consider how to derive asymptotic expansions for some special integrals depending on parameters. These occur in many applications. We shall begin to demonstrate simple but useful methods by means of the example of the incomplete gamma function.

3 Method of integration by parts

Incomplete gamma function

The incomplete gamma function

$$\gamma(a, x) = \int_0^x e^{-s} s^{a-1}\, ds \tag{13}$$

plays an important role in mathematical physics. We would like to obtain simple formulas with which to calculate $\gamma(a, x)$ for positive x and fixed positive a. The function $\gamma(a, x)$ is "a portion" of the Gamma function

$$\Gamma(a) = \int_0^\infty e^{-s} s^{a-1}\, ds$$

for which, as is well known, $\Gamma(n) = (n-1)!$.

We will demonstrate the two simplest but frequently used methods of getting asymptotics. One uses well known Taylor expansions and the other is the method of integration by parts. The two asymptotics which we shall obtain are used in different ranges of the variable x.

To find an asymptotic expansion of $\gamma(a, x)$ for small x, we can use the expansion

$$e^{-s} = \sum_{k=0}^{\infty} \frac{(-1)^k s^k}{k!}.$$

Substituting this into (13) and integrating term-wise we get

$$\gamma(a, x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{a+k}}{(a+k)k!}.$$

This is a convergent series whose radius of convergence is infinite. Thus we could use this formula to calculate $\gamma(a, x)$ at any x. However, for large x we need to take many terms to achieve an accurate result. For large x we will find an asymptotic expansion of a supplementary function

$$\Gamma(a, x) = \Gamma(a) - \gamma(a, x) = \int_x^\infty e^{-s} s^{a-1}\, ds;$$

this function is important in itself, since by a change of variable we can see that the error integral $\mathrm{Erfc}(x)$ has representation

$$\mathrm{Erfc}(x) = \int_x^\infty e^{-t^2}\, dt = \frac{1}{2}\int_{x^2}^\infty e^{-s} s^{-1/2}\, ds = \frac{1}{2}\Gamma\left(\frac{1}{2}, x^2\right). \qquad (14)$$

It is seen that the integral in $\Gamma(a, x)$ is convergent. We will obtain its asymptotic expansion for large x. Integrating by parts we have

$$\int_x^\infty e^{-s} s^{a-1}\, ds = e^{-x} x^{a-1} + (a-1)\int_x^\infty e^{-s} s^{a-2}\, ds.$$

Let us repeat the procedure with respect the integral on the right:

$$\int_x^\infty e^{-s} s^{a-1}\, ds = e^{-x} x^{a-1} + (a-1)e^{-x} x^{a-2} + \\ + (a-1)(a-2)\int_x^\infty e^{-s} s^{a-3}\, ds.$$

It is clear that many-fold repetition of the procedure gives us

$$\int_x^\infty e^{-s}s^{a-1}\,ds = e^{-x}[x^{a-1} + (a-1)x^{a-2} + \cdots + \\ + (a-1)(a-2)\cdots(a-n+1)x^{a-n}] + \\ + (a-1)(a-2)\cdots(a-n)\int_x^\infty e^{-s}s^{a-n-1}\,ds. \quad (15)$$

It is seen that when $x \to \infty$ the sequence $\{e^{-x}x^{a-k}\}$ is asymptotic. Let us verify that the last term is $O(e^{-x}x^{a-n-1})$ as $x \to \infty$. Indeed

$$\int_x^\infty e^{-s}s^{a-n-1}\,ds \le x^{a-n-1}\int_x^\infty e^{-s}\,ds = e^{-x}x^{a-n-1},$$

which means that we really have an asymptotic expansion of $\Gamma(a,x)$ for $x \to \infty$.

By the properties of $\Gamma(a)$ we can rewrite (15) as

$$\Gamma(a,x) \sim e^{-x}\sum_{k=1}^\infty \frac{\Gamma(a)}{\Gamma(a-k+1)}x^{a-k} \quad \text{as } x \to \infty.$$

For large $x > 0$ this expansion gives accurate results for a small number of terms. From this and (14) we derive an important expansion

$$\operatorname{Erfc}(x) = \frac{1}{2}\Gamma\left(\frac{1}{2}, x^2\right) \sim \frac{1}{2}e^{-x^2}\Gamma\left(\frac{1}{2}\right)\sum_{k=1}^\infty \frac{1}{x^{2k-1}\,\Gamma\left(\frac{3}{2}-k\right)} \\ = \frac{1}{2}\sqrt{\pi}e^{-x^2}\sum_{k=1}^\infty \frac{1}{x^{2k-1}\,\Gamma\left(\frac{3}{2}-k\right)} \quad \text{as } x \to \infty.$$

As with other expansions of this type, computations are best done with one or two terms — despite the series notation. Many asymptotic expansions are actually divergent when considered as series. So adding more and more terms *will*, beyond a certain number of terms, lead to deterioration in the approximation. However, for a fixed number of terms the approximation always improves as the independent variable $x \to x_0$.

The Fourier integral

Let us consider an asymptotic expansion for large ξ for the Fourier integral on a finite interval

$$\int_a^b e^{i\xi x}\varphi(x)\,dx$$

when $\varphi(x)$ has n continuous derivatives on $[a,b]$. Integration by parts gives us

$$\int_a^b e^{i\xi x}\varphi(x)\,dx = -\frac{1}{(i\xi)}\left[e^{i\xi a}\varphi(a) - e^{i\xi b}\varphi(b)\right] - \frac{1}{(i\xi)}\int_a^b e^{i\xi x}\varphi'(x)\,dx.$$

A second integration by parts yields

$$\int_a^b e^{i\xi x}\varphi(x)\,dx = -\frac{1}{(i\xi)}\left[e^{i\xi a}\varphi(a) - e^{i\xi b}\varphi(b)\right] + \\ + \frac{1}{(i\xi)^2}\left[e^{i\xi a}\varphi'(a) - e^{i\xi b}\varphi'(b)\right] + \\ + \frac{1}{(i\xi)^2}\int_a^b e^{i\xi x}\varphi''(x)\,dx.$$

Continuing in this way, after n-fold integration by parts we have

$$\int_a^b e^{i\xi x}\varphi(x)\,dx = \sum_{k=0}^{n-1}\frac{(-1)^{k+1}}{(i\xi)^{k+1}}\left[e^{i\xi a}\varphi^{(k)}(a) - e^{i\xi b}\varphi^{(k)}(b)\right] + \\ + \frac{(-1)^n}{(i\xi)^n}\int_a^b e^{i\xi x}\varphi^{(n)}(x)\,dx.$$

The last integral term can be estimated as

$$\left|\frac{(-1)^n}{(i\xi)^n}\int_a^b e^{i\xi x}\varphi^{(n)}(x)\,dx\right| = \frac{1}{|\xi|^n}\left|\int_a^b e^{i\xi x}\varphi^{(n)}(x)\,dx\right| = o\left(\frac{1}{|\xi|^n}\right)$$

since by a calculus theorem $\int_a^b e^{i\xi x}\varphi^{(n)}(x)\,dx \to 0$ as $|\xi| \to \infty$. Thus we have

$$\int_a^b e^{i\xi x}\varphi(x)\,dx = \sum_{k=0}^{n-1}\frac{i^{k+1}}{\xi^{k+1}}\left[e^{i\xi a}\varphi^{(k)}(a) - e^{i\xi b}\varphi^{(k)}(b)\right] + o\left(\frac{1}{|\xi|^n}\right),$$

which defines an asymptotic expansion of the Fourier integral as $|\xi| \to \infty$.

Exercise 3.1. Let $a > 0$. Use the method of integration by parts to demonstrate that the asymptotic expansion

$$\int_x^\infty \frac{e^{is}}{s^a}\,ds \sim \frac{ie^{ix}}{x^a}\sum_{k=0}^{\infty}\frac{\Gamma(a+k)}{\Gamma(a)(ix)^k}$$

is valid as $x \to \infty$. Note that the Fresnel integrals

$$\int_x^\infty \cos(s^2)\,ds, \qquad \int_x^\infty \sin(s^2)\,ds,$$

used in optics are a particular case (the real and imaginary parts) of the above integral when $a = 1/2$.

4 Equations depending on a parameter analytically

The implicit function theorem of calculus is well known. We state it in a form convenient for obtaining an asymptotic expansion of a solution with respect to a parameter ε.

Theorem 4.1. Suppose $f(x_0, 0) = 0$, the function $y = f(x, \varepsilon)$ is continuous in both variables in some neighborhood of $(x_0, 0)$, $\partial f(x, \varepsilon)/\partial x$ is continuous in the same neighborhood of $(x_0, 0)$, and $\partial f(x, \varepsilon)/\partial x \neq 0$ at point (x_0, ε). Then there is a neighborhood $\varepsilon < \varepsilon_0$ and $|x - x_0| < \rho$ in which there exists a unique solution $x = x(\varepsilon)$ of the equation $f(x, \varepsilon) = 0$; the function $x = x(\varepsilon)$ is continuous at the point $\varepsilon = 0$.

Less familiar is the theorem on the analytic dependence of the same solution on a parameter. Let us add to the conditions of Theorem 1 that the function $y = f(x, \varepsilon)$ is holomorphic at point $(x_0, 0)$: i.e., there is a neighborhood $\varepsilon < \varepsilon_0$ and $|x - x_0| < \rho_0$ in which the double power series

$$f(x, \varepsilon) = \sum_{k+m \geq 1} f_{km}(x - x_0)^k \varepsilon^m$$

is convergent (note that $f_{00} = 0$ since $f(x_0, 0) = 0$). It appears that in this case, in some neighborhood of the zero function $x = x(\varepsilon)$ is holomorphic, which means that it has the form

$$x = x_0 + \sum_{k=1}^{\infty} x_k \, \varepsilon^k \tag{16}$$

where the series has nonzero radius of convergence.

In applications it is quite frequent that the equation $x = x(\varepsilon)$ has the form of a polynomial in two variables, or that the variables are under trigonometric or exponential functions so the last condition is fulfilled automatically. This means that, wishing to find the dependence of a solution on a parameter in such a case, we can use quite a simple technique. We represent the equation in the power form

$$\sum_{k+m \geq 1} f_{km}(x - x_0)^k \varepsilon^m = 0$$

and substitute (16) into it. Then, collecting coefficients of like powers of ε and equating them to zero, we obtain equations for finding the coefficients x_k. The equation at ε gives us x_1, the next equation allows us to calculate x_2, and so on. At each step we obtain a linear equation with respect to the next variable, so this calculation is quite easy.

This theorem and the method can be extended to operator equations of the form

$$F(x, \varepsilon) = 0,$$

where x and ε belong to some Banach spaces and the image of F is also a Banach space. The form of the result simply copies the above but of course, it requires harder techniques to apply.

If $F(x, \varepsilon)$ is continuous in some neighborhood of a point $(x_0, 0)$ at which $F(x_0, 0) = 0$ together with its partial derivative $F_x(x, \varepsilon)$, there exists a continuous inverse of $F_x(x, \varepsilon)$ at point $(x_0, 0)$ and $F(x, \varepsilon)$ is holomorphic in the same point, which means that in some neighborhood of $(x_0, 0)$ there is a convergent series

$$F(x,\varepsilon) = \sum_{k+m\geq 1} F_{km}(x-x_0)^k \varepsilon^m$$

where the operator coefficients F_{km} are continuous in zero and homogeneous in their variables, that is

$$F_{km}(\alpha(x-x_0))^k(\beta\varepsilon)^m = \alpha^k\beta^m F_{km}(x-x_0)^k(\varepsilon)^m$$

for any numbers α, β. Then there is a neighborhood of point $(x_0, 0)$ in which there is a unique solution of the equation $F(x,\varepsilon) = 0$ that is holomorphic in ε in some neighborhood of zero, so there is a series representation

$$x = x_0 + \sum_{k=1}^{\infty} x_k \varepsilon^k$$

where $x_k(\alpha\varepsilon)^k = \alpha^k x_k \varepsilon^k$ for any number α that converges uniformly in this neighborhood.

This result, given without proof, allows us to find an asymptotic dependence of solution of many boundary value problems when the equations and boundary conditions depend on unknown functions and parameters analytically. Let us demonstrate two examples.

1. A nonlinear integral equation. Let us consider a nonlinear integral equation of the Fredholm type:

$$u(t) - \int_a^b K(t,s,u(s),\varepsilon)\,ds = f(t), \tag{17}$$

where $u = u(t)$ is an unknown function on a finite segment $[a,b]$ and ε is a small parameter. Suppose that the kernel $K(t,s,u,\varepsilon)$ is a continuous function of all its arguments when $t,s \in [a,b]$, $-\infty < u < \infty$, $|\varepsilon| < \rho_0$. Let its particular derivative $K_u(t,s,u,\varepsilon)$ be uniformly continuous on the same set, $f = f(t)$ continuous on $[a,b]$, and suppose that at $\varepsilon = 0$ there is a solution of $u_0(t)$ of (17) that is continuous on $[a,b]$. Suppose that $K(t,s,u,\varepsilon)$ is analytic with respect to u and ε.

At last let us introduce the partial derivative of the operator of the problem for which we need to study the problem of the inverse:

$$F_u(u_0,0)v = v(t) - \int_a^b K_u\,(t,s,u_0(s),0)\,v(s)\,ds.$$

We consider this problem in the space $C(a,b)$. By the above result if we know that the operator $F_u(u_0,0)$ has a continuous inverse then equation (17) has an analytic solution representable in a series form:

$$u(t,\varepsilon) = \sum_{k=0}^{\infty} u_k(t)\,\varepsilon^k. \tag{18}$$

It can be shown that under the above conditions the integral operator is compact in $C(a,b)$. So to study the inverse to $F_u(u_0,0)$ we can appeal to the Fredholm–Schauder–Riesz theory. Suppose that the equation with respect to $v=v(t)$,

$$v(t)-\int_a^b K_u\left(t,s,u_0(s),0\right)v(s)\,ds=0, \tag{19}$$

has only the trivial solution $v=0$. This means that considering this as a particular case of the eigenvalue problem

$$v(t)-\lambda\int_a^b K_u\left(t,s,u_0(s),0\right)v(s)\,ds=0$$

with $\lambda=1$ we have that it is not an eigenvalue. By the Fredholm–Schauder–Riesz theory this means that this operator has a continuous inverse in $C(a,b)$.

Thus we can explore the small parameter method to find the dependence of $u(t,\varepsilon)$ on ε, that is, we need to substitute (18) into equation (17) where kernel $K(t,s,u,\varepsilon)$ is expanded into power series. Equating the coefficients at equal powers of ε we will have the system of linear Fredholm equations with respect to the coefficients $u_k(t)$. This system is recurrent: finding the first k unknown functions, on the next step we get an equation in a single unknown.

2. A boundary value problem. Suppose we need to find a solution of a boundary value problem depending on a small parameter ε:

$$y''=f(x,y(x),y'(x),\varepsilon), \tag{20}$$

$$y(0)=g_0(\varepsilon), \qquad\qquad y(1)=g_1(\varepsilon). \tag{21}$$

Suppose that $f(x,y,v,\varepsilon)$ is continuous in the area $x\in[0,1]$, $-\infty<x,y<\infty$, $|\varepsilon|<\rho$ and that the partial derivatives $f_y(x,y,v,\varepsilon)$ and $f_v(x,y,v,\varepsilon)$ are uniformly continuous in the same range. Finally, let us suppose that $f_y(x,y,v,\varepsilon)$ is analytic with respect to y,v,ε.

Knowing the Green function that is a solution of the problem

$$y''(x)=\delta(x-x_0), \qquad y(0)=0=y(1),$$

we can easily reduce this problem to the previous integral equation, which means that we immediately have the needed result if we know that the corresponding equation (19) has only the zero solution. Understanding this, we mention that it is not really necessary to reduce the problem to the integral form, but we can formulate everything in the initial terms. Let $y_0(x)$ be a solution of (20) at $\varepsilon=0$. Instead of the integral equation (19) we should consider the boundary value problem

$$\begin{aligned} z''(x)&=f_y(x,y_0(x),y_0'(x),0)z(x)+f_{y'}(x,y_0(x),y_0'(x),0)z'(x),\\ z(0)&=0=z(1), \end{aligned} \tag{22}$$

with respect to $z = z(x)$. If this equation has the only solution $z(x) = 0$ we have a solution to (20) at small ε has a form

$$y(t, \varepsilon) = \sum_{k=0}^{\infty} y_k(t)\, \varepsilon^k \tag{23}$$

that can be found by successive solution of boundary value problems that arise when we substitute (23) into (20) where f is written in the series form and then equate the coefficients at the same powers of ε.

Exercise 4.1. Find a series dependence of solutions of the following equations:

(a) $x^2 - 2x + \varepsilon = 0$

(b) $x = \cos(x + \varepsilon)$

5 Branching of solutions

Before studying a general problem it is often advantageous to consider a simple problem that has all the necessary features. We would like to study a case of the previous section when f_x has no inverse at a solution for $\varepsilon = 0$. A simple example of such a problem is the problem of dependence on ε of the solution of an equation:

Problem 5.1. Find the asymptotics of solution of $x^2 = \varepsilon$ as $\varepsilon \to 0$.

For this equation $x = 0$ is a solution when $\varepsilon = 0$, and $f(x, \varepsilon) = x^2 - \varepsilon$, being analytic, does not satisfy the condition $f_x(0, 0) \neq 0$, so we cannot say there is an analytic dependence of x on ε.

But now we can easily solve the equation. Its solutions are $x = \pm\sqrt{\varepsilon}$. First of all we see the non-uniqueness of solution. Secondly, if we wish to find real roots they exist only for $\varepsilon > 0$. (This situation is typical for many practical problems when increasing some parameters, the amplitude of a force or voltage or something else, we get some point from which there go several branches of possible change of dependence of a solution.) Thirdly, the point $\varepsilon = 0$ is a point of branching of the function $y = \sqrt{\varepsilon}$, so there is no integer power series that represents it. However there is an expansion in power series with respect to $\varepsilon^{1/2}$ (of course, it is funny to call it an expansion since it is only one term of an expansion, however we discuss all this knowing what we get in more complex situations, that is why we direct the reader's attention to this fact).

Note that changing ε to ε^2 we could avoid square roots and get usual power series. This is common for many practical problems: an appropriate change of a parameter brings power asymptotics of a solution.

Let us consider another problem.

Problem 5.2. Find the asymptotics of solution of $\varepsilon x^2 - x = 0$ as $\varepsilon \to 0$.

This problem has two solutions: $x = 0$, $x = 1/\varepsilon$. We see that now a quadratic equation has roots one of which is impossible to approximate putting $\varepsilon = 0$. In this case $\varepsilon = 0$ is a pole of the latter solution.

Now we begin to study an equation $f(x, \varepsilon) = 0$ that has a solution $x = 0$ at $\varepsilon = 0$ when the function is analytic in both variables but $f_x(0, 0) = 0$ wishing to get a dependence $x = x(\varepsilon)$ at small ε. For simplicity we will consider a case when

$$f(x, \varepsilon) = \sum_{k=0}^{n} a_k(\varepsilon) x^k. \tag{24}$$

Suppose that each a_k is presented in some neighborhood of $\varepsilon = 0$ as a convergent series

$$a_k(\varepsilon) = \varepsilon^{p_k} \sum_{n=0}^{\infty} a_{kn} \varepsilon^{n/N}. \tag{25}$$

We suppose that if $a_k(\varepsilon) \neq 0$ then $a_{k0} \neq 0$ and p_k are rational numbers and N is an integer.

An equation of the type (24) is one to which many problems of mathematical physics on branching of solutions reduce. We note that quite frequently the original problem has some parameter under the symbol of some root, in this case the change of parameter brings one to the representation (25). We describe only the technique how to find the asymptotics of the roots of (24) that descends to Newton. Suppose the principal term of the asymptotics is $x_0 \varepsilon^r$ so

$$x(\varepsilon) = x_0 \varepsilon^r + o(\varepsilon^r) \quad \text{as } \varepsilon \to 0. \tag{26}$$

Let us substitute it into (24) and collect terms with equal powers of ε. It is clear that if we wish the term of lowest degree to be cancelled we need to have at least two terms with the lowest degree of ε. This brings us to a practical way of finding r, the degree in (26). It is easy to see that the lowest degree of ε in (24) is among the following terms:

$$a_{00}\varepsilon^{p_0},\ a_{10}\varepsilon^{p_1+r},\ a_{20}\varepsilon^{p_1+2r}, \ldots, a_{n0}\varepsilon^{p_1+nr}$$

We recall that some of the a_{k0} are zeros. For practical finding such variants for r when several degrees $p_k + kr$ are equal and corresponding numbers k, Newton proposed to construct a diagram as follows. We need to find k and m such that $p_k + kr = p_m + mr$ and for all s with $a_{s0} \neq 0$ there holds $p_s + sr \geq p_k + kr$.

Take a cartesian coordinate plane and check points (k, p_k) for nonzero a_{k0}. Suppose that $a_{0,0} \neq 0$. Take a ray beginning at $(0, p_0)$ codirected with negative direction of y-axis and rotate it counterclockwise until it meets a point (k_1, p_{k_1}). On the direct line through $(0, p_0)$ and (k_1, p_{k_1}) there can be other points. It is clear that taking $r_0 = (p_{k_1} - p_0)/k_1$ we get the needed equal degrees $p_0 = p_{k_1} + k + 1r_0$. Fix this direct line. If we draw direct lines parallel to this one through other checked points they will lie above this one that means that $p_s + sr_0 \geq p_0$.

Substituting (26) with $r = r_0$ into (24) we can find the coefficient x_0 of the asymptotics (26) that is nonunique if there are more than two checked points on the line through $(0, p_0)$ and (k_1, p_{k_1}).

Let us find other possible values for r in (26). For this take the farthest from $(0, p_0)$ checked points on the line through $(0, p_0)$ and (k_1, p_{k_1}), let it be (m, p_m) and rotate counterclockwise the ray from the old direction until it meets the next checked point (t, p_t). Then the second possible value of r is $r_1 = (p_t - p_m)/(t - m)$. Here we also can find the principal term of another asymptotic (of another root(s)). By the same reasons as above for the other checked points we will have $p_k + kr_1 \geq p_m mr_1$. In a similar fashion we can find new roots (their principal members).

Repeating the procedure for the farthest from (m, p_m) point on the line through (m, p_m) and (t, p_t), we find a new chain of the Newton diagram.

Note that if corresponding r_k is positive then the solution is continuous at $\varepsilon = 0$, otherwise it tends to infinity.

If we would like to find higher terms of the asymptotics we should substitute

$$x(\varepsilon) = x_{0k}\varepsilon^{r_k} + x_{1k}\varepsilon^{r} + o(\varepsilon^{r}), \qquad r > r_k$$

into (24) and repeat the procedure.

In this way we can find asymptotics of any length

$$x(\varepsilon) \sim x_{0k}\varepsilon^{r_k} + x_{1k}\varepsilon^{r_{k_1}} + x_{2k}\varepsilon^{r_{k_2}} + \cdots .$$

This is called the Puiso expansion.

It is shown that under some conditions the degrees r_{k_s} are of the form $r_{k_s} = (q + s)/Q$ where q, Q are fixed integers. This happens to many important practical problems whose asymptotics are power series in $\varepsilon^{1/2}$ or $\varepsilon^{1/3}$.

I forgot to introduce the definition of a branching point of an equation. It is a solution in any neighborhood of which and some small ε there are two solutions of the equation. Thus when we find positive r_k we find a point of branching of the equation. If it is negative, it is not a point branching since another solution tends to infinity as $\varepsilon \to 0$.

It is quite easy to extend the theory to equations $f(x, \varepsilon) = 0$ with an analytic function with respect to x and ε. In this case we can find continuous branching of solutions that can be done quite similar to the above. Now the Newton diagram would be infinite but the range where r_k are positive is always finite so we can apply the same techniques to this.

The theory of asymptotics of an algebraic equations is extended to equations in Banach spaces. When it is a nonlinear equation with a parameter having branching points, the use of the Lyapunov–Schmidt method allows us to reduce the problem in the neighborhood of such a point to a finite dimensional equation that can be solved by the techniques similar to the above.

6 Ordinary differential equations with singular points

Sometimes we are interested in the behavior of solution to a Cauchy problem on a very large range $0 < t < T$:

$$y'(t) = f(y(t)), \qquad y(0) = y_0.$$

In particular, this class of problems includes the problem of motion of our Earth (when $y = y(t)$ is a vector function). We know that numerical calculations do not give an accurate result quite frequently because of accumulated errors. We can change the variable $t \mapsto Tt$. Denoting $\varepsilon = 1/T$ we get a problem

$$\varepsilon\, y'(t) = f(y(t)), \qquad y(0) = y_0, \tag{27}$$

which we would like to solve for small $\varepsilon > 0$ on $[0, 1]$.

If we put $\varepsilon = 0$ in (27) formally we get a non-differential equation $f(y(t)) = 0$ that in practical cases has only few separate solutions that means that we cannot satisfy the initial condition $y(0) = y_0$. What we can tell about solutions of similar equations? Let us consider this question for a simple example which we can solve analytically:

$$\varepsilon\, y'(t) + y(t) = 1, \qquad y(0) = 0 \tag{28}$$

when ε is positive and small. The equation $f(y(t)) = 0$ reduces to $y(t) = 1$. However the solution to (28) is

$$y = -e^{-t/\varepsilon} + 1.$$

This solution contains two terms, one of which, $y = 1$ is called regular part of solution and another, $y = -e^{-\frac{t}{\varepsilon}}$ the boundary layer solution. The smaller is ε the smaller is the domain $[0, t_0]$ where we need to take into account this term calculating equation numerically. The name 'boundary layer' came from hydrodynamics where they studied the problems whose solutions had similar behavior near the boundary of a liquid: there is fast change in values near the boundary and outside some small neighborhood of the boundary the solution is quite regular. Similar parts of solutions are in electricity, the corresponding part of solution is called the transient process that Mr Mike knows better than me.

If we can find an asymptotics near the boundary exactly (or numerically) then the regular part of solution, as a rule can be found numerically quite accurately.

We can consider ODEs of the type

$$g(x)y'(x) = f(y(x))$$

where there is a point x_0 such that $g(x_0) = 0$. In this the equation also has a singular point at which the solution behaves in a singular way.

There is a theory of equations of these types and the methods how to find their asymptotics.

Quite similar situation with boundary layers happens to many problems: There are partial differential equations with small parameters at the highest derivatives. Their theory is hard but useful.

Not to think that the situation of equation (28) is the most common we can consider the problem

$$\varepsilon y'(t) - y(t) = 1, \qquad y(0) = 0,$$

with small positive ε. Now a solution is

$$y = e^{t/\varepsilon} - 1$$

and the fast growing part penetrates everywhere. This problem relates to incorrect problems. One of the frequently met equations with a small parameter at the highest derivative is one of the form:

$$\varepsilon^2 y''(x) + y(x) = \varepsilon f(x, y(x), \varepsilon)$$

and its variants. This equation can be met in boundary value problems and Cauchy problems. The main part of the main term of the asymptotics of its solution is a fast oscillating function $a \sin(x/\varepsilon + \varphi)$. There are many methods that in much relate to averaging how to find the main asymptotics of the solution. Take into account that the problem of 3 bodies (it is when they move being mutually attracted by the gravity law) is of this type. So we wanted to mention that boundary layers is not the only what can be met in this theory.