Convergence in Mechanics

L.P. Lebedev

Department of Mechanics and Mathematics
Rostov State University, Russia
and
Departamento de Matemáticas
Universidad Nacional de Colombia

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Abstract

Many results of mechanical engineering (and other areas of engineering and mathematics) are approximate; they are calculated using general computer implementations of the finite element method (FEM). Some results take the form of series expansions. Sometimes the problem of convergence of such series or approximations can be studied using the simple tools of calculus. On other occasions the analysis can be quite complicated, requiring us to apply methods from functional analysis. We would like to shed some light on the nature of convergence, especially as it relates to the practical problems of mechanics. Of course, convergence in mechanics does not differ from convergence in pure mathematics; however, within the framework of mechanics we can draw on examples in order to illustrate the importance of the topic. We begin with definitions and simple examples.

Classical convergence and the finite element method

The concept of convergence was given a strict definition in mathematics not so long ago (on a historical time scale). It involves the "ε–δ language" originated by the French mathematician Augustin Cauchy (1789–1857). Before Cauchy's time, justifications of convergence had to be provided in an *ad hoc* manner. To make matters worse, the reasoning used was often no more rigorous than the statements we sometimes hear from present-day engineers: "We first compute our result to within a certain accuracy, then recompute it to a better degree of accuracy. If we compare the results and find that they agree to three significant decimal digits, then we know convergence has been reached."

As with many other concepts in mathematics, the notion of sequence convergence began its life on an intuitive basis. The core idea is, of course, that a point a is called the limit of a sequence $\{x_n\}$ if the points x_n approach a more and more closely as n gets larger and larger. This "definition" works quite well when $\{x_n\}$ is a monotonic numerical sequence. If the behavior of $\{x_n\}$ is otherwise, however, we run into immediate difficulties: we cannot check each term of the sequence for closeness to the limit since this would require us to perform an infinite number of comparisons. Cauchy proposed a definition of convergence that should be familiar to every student:

Definition 1. A sequence $\{x_n\}$ converges to a number a if for any $\varepsilon > 0$ there is an integer N, dependent on ε, such that for $n > N$ we have $|x_n - a| < \varepsilon$. In this case a is called the limit of $\{x_n\}$. We write $\lim_{n\to\infty} x_n = a$, or $x_n \to a$ as $n \to \infty$.

Hence to prove that a is the limit of $\{x_n\}$, we must verify this definition for *all*[1] the values $\varepsilon > 0$; on the other hand, to show that a is *not* the limit we can merely display one $\varepsilon_0 > 0$ for which there is no corresponding N as required by the definition.

Does Cauchy's definition completely rule out the possibility that we might have to perform an infinite number of checks on the convergence process? Not if the sequence behaves in a persistently irregular fashion. But if $\{x_n\}$ is given by a convenient formula or can be well approximated by one (which is often the case when the sequence arises from geometrical or physical considerations), then the necessary check reduces to the solution of the inequality $|x_n - a| < \varepsilon$ with respect to n. If for any $\varepsilon > 0$ its solution set contains some interval $[N, \infty)$ (with $N = N(\varepsilon)$), then a is the required limit; otherwise it is not. Of course, any elementary calculus textbook contains many simple examples of this type.

In mechanics, besides sequences of real numbers we work with sequences of vectors and tensors. For these it is also desirable to have a notion of convergence. We assume that the reader has met such quantities in classical mechanics and elsewhere, and is therefore aware of at least some of their properties. In particular, the set of all vectors (or tensors) of the same (finite) dimension, when endowed with the usual algebraic operations of addition and multiplication by real (or complex) numbers, is a vector space. Let $\{\mathbf{x}_k\}$ be a sequence of vectors or tensors. The definition that extends Cauchy's definition for numerical sequences is

Definition 2. We say that $\mathbf{a}$ is the limit of the sequence $\{\mathbf{x}_k\}$ if for any $\varepsilon > 0$ there is an integer N (dependent on ε) such that for all $k > N$ the inequality $\|\mathbf{x}_k - \mathbf{a}\| < \varepsilon$ holds.

We note one important modification that has been made: the absolute value symbol of Definition 1 has been changed to the more general norm symbol of Definition 2. Let us therefore briefly review concept of a norm on a vector space. The idea is simple: to every vector (or tensor) $\mathbf{x}$ there corresponds a number

[1]Of course, it suffices to verify it for small values of ε; as soon as we find N for some ε_0, then this same N could be used for any ε greater than ε_0. So we need not check the definition for all values of $\varepsilon > 0$, but can take a monotonic sequence $\{\varepsilon_k\}$ tending to zero.

denoted as $\|\mathbf{x}\|$ that possesses the following properties. If $\|\mathbf{x}\|$ and $\|\mathbf{y}\|$ are any vectors (or tensors) and λ is any real number (or complex number, depending on the type of vector space), we have

1. $\|\mathbf{x}\| \geq 0$, with $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = 0$;
2. $\|\lambda \mathbf{x}\| = |\lambda| \|\mathbf{x}\|$;
3. $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

These properties are known as the norm axioms.

The norm measures the "magnitude" of an element in the space, and the quantity $\|\mathbf{x} - \mathbf{y}\|$ is a natural measure of distance between the two elements $\mathbf{x}$ and $\mathbf{y}$. For a vector given in a Cartesian frame as $\mathbf{x} = (x_1, x_2, \ldots, x_n)$, the familiar Euclidean norm is given by

$$\|\mathbf{x}\| = \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{1/2}.$$

This norm is related to the ordinary vector dot product through the relation $\|\mathbf{x}\| = (\mathbf{x} \cdot \mathbf{x})^{1/2}$, and in elementary mathematics textbooks it is the only norm used in conjunction with n-dimensional vector spaces. In mathematics there is a strong preference for the use of dimensionless quantities, and the prevalence of numerical computations has brought a bit of this practice into mechanics. However, in mechanics one must often work with quantities that are not dimensionless; these occur naturally in practice and, moreover, can be useful since they allow us to perform rough (at least order-of-magnitude) checks on manual calculations. In such a situation, where the vectors carry units, the norm on the space will also carry units (e.g., Newtons). Some of the vectors utilized in mechanics can seem a bit strange; we can, for example, collect the components of the position vector of a point along with the impulse vector acting at the same point, and declare the result to be a point in a six-dimensional space. In this case we must introduce additional coefficients into the form of the Euclidean norm; otherwise we would produce senseless expressions in which quantities having unlike units would have to be added. Even a simple change in units for one of the components will bring some additional coefficients into the Euclidean norm. In this way we see that the expression

$$\|\mathbf{x}\| = \left(k_1 x_1^2 + k_2 x_2^2 + \cdots + k_n x_n^2\right)^{1/2},$$

with constants $k_i > 0$, could play the role of the norm on the same vector space when the vectors are given initially in a Cartesian frame. It is easy to verify that all the norm axioms hold for this expression, and hence it really is a norm.[2] So we see that the norm on a vector space is not unique in general. In fact, it turns out that there are infinitely many norms that can be imposed on a given

[2] An extremely common error among novices is to denote some quantity in a standard way (as we have denoted an expression as a norm here) and then to use all the properties that follow from the definition of this standard item. The remedy is, of course, to actually verify all the desired properties. Only then can one be certain that the title truly fits.

space. Any of these will provide some characterization of the size of a vector (along with the distance between vectors). The specific choice of norm depends on the goals we wish to pursue in our analysis.

Other types of norms find application with n-dimensional spaces. One of these is

$$\|\mathbf{x}\|_p = \left(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\right)^{1/p},$$

where p is a fixed real number satisfying $p \geq 1$. Another is

$$\|\mathbf{x}\|_m = \max\{|x_1|, |x_2|, \ldots, |x_n|\}.$$

Quite conveniently, it turns out that if a sequence $\{\mathbf{x}_n\}$ in n-dimensional space converges to $\mathbf{a}$ with respect to one of its norms, then it converges to the same limit $\mathbf{a}$ with respect to any other norm. This follows from the equivalence of all norms over an n-dimensional space, a fact which we formulate as follows:

Lemma 1 (norm equivalence). Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms over an n-dimensional space X. Then there are two positive constants m and M such that for any $\mathbf{x} \in X$ the inequality

$$m \leq \frac{\|\mathbf{x}\|_1}{\|\mathbf{x}\|_2} \leq M \tag{1}$$

holds.

It is clear that the lemma follows from the particular case when $\|\cdot\|_1$ is the Euclidean norm. We propose that the reader supply the details of the proof.

Thus, in a finite-dimensional space, convergence of a sequence of vectors with respect to all norms is the same; for this reason we do not distinguish between n-dimensional vector spaces based upon differences in their norms. In particular, it is sufficient to test for convergence with respect to the Euclidean norm. Equivalently, we can test the n sequences composed of the individual vector components: if they all converge, then the vector sequence converges; if at least one of them diverges then the vector sequence diverges, and vice versa.

Now let us move on to the problem of convergence of infinite-dimensional vectors. Vectors with countably many components arise when we consider any sequence as a vector whose components are the members of the sequence. They also arise when we consider the series expansion of a function: that is, the coefficients obtained by expanding a given function into a Fourier or Taylor series can be regarded as a vector having countably many components. On the set of such vectors, considerations of the norm and of convergence with respect to that norm are not as simple as they are for a finite-dimensional vector space. The norms for the finite-dimensional case can be extended to the infinite-dimensional case, for example, as follows:

$$\|\mathbf{x}\|_p = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p} \quad \text{for some } p \geq 1,$$

$$\|\mathbf{x}\|_m = \sup_k \{|x_1|, |x_2|, \ldots\}.$$

But we do begin to notice departures from the finite-dimensional case. First, we recall that the norm of any element must be a finite nonnegative number. If we consider a vector $\mathbf{x}_0$ having infinitely many components and whose kth component is $1/k$, we see that $\|\mathbf{x}_0\|_m = 1$; on the other hand, $\|\mathbf{x}\|_p$ for $p = 1$ (we will denote this as $\|\mathbf{x}\|_1$) is infinite because the series $\sum_{k=1}^{\infty} 1/k$ diverges. Under the norms $\|\cdot\|_m$ and $\|\cdot\|_1$ then, the sets of infinite-dimensional vectors are different normed spaces. The failure of Lemma 1 to extend to the infinite-dimensional case means, of course, that all types of sequence convergence are not equivalent in this case. In particular, component-wise convergence differs crucially from convergence with respect to the norm $\|\cdot\|_2$. If we take the sequence $\{\mathbf{x}_k\}$ of elements $\mathbf{x}_k = (x_{k1}, x_{k2}, \ldots)$ such that

$$x_{ki} = \begin{cases} 1, & i = k, \\ 0, & i \neq k, \end{cases}$$

then for every fixed i we have $\lim_{k\to\infty} x_{ki} = 0$. Therefore this sequence tends to zero component-wise,[3] whereas with respect to the norm $\|\cdot\|_2$ it cannot converge because $\|\mathbf{x}_n - \mathbf{x}_m\|_e = \sqrt{2}$ when $n \neq m$. Thus, for an infinite-dimensional vector space a change in the norm can determine a different set of elements or a different type of sequence convergence. Various choices of normed space can make sense, depending upon the problem under consideration. In mechanics one popular choice is the space ℓ^2, which consists of all the infinite-dimensional vectors $\mathbf{x}$ that have finite norm $\|\mathbf{x}\|_2$. In this space we can introduce an inner product just as we can in an n-dimensional Euclidean space.[4]

Next, let us consider the problem of convergence of functions. We begin with functions that are continuous on a bounded closed set $D \subset \mathbb{R}^n$. We denote by $C(D)$ the normed space consisting of all such functions along with the norm

$$\|f(x)\|_C = \max_{x\in D} |f(x)|.$$

In particular, when $D = [0,1]$ we write $C(0,1)$. Since $C(D)$ is an infinite-dimensional space, we cannot expect that simple pointwise convergence of a sequence $\{f_n(x)\} \subset C(D)$ will guarantee the existence of a limit function from $C(D)$. Sufficient conditions are, however, given by the following famous result.

Theorem 1 (Weierstrass). Suppose $\{f_n(x)\} \subset C(D)$ is a Cauchy sequence in $C(D)$; that is, suppose we have

$$\max_D |f_n(x) - f_m(x)| \to 0 \quad \text{as } m, n \to \infty.$$

Then there exists $f(x) \in C(D)$ to which the sequence converges uniformly:

$$\max_D |f_n(x) - f(x)| \to 0 \quad \text{as } n \to \infty.$$

[3] This is the so called weak convergence of the sequence to zero in the space ℓ^2.

[4] More importantly, through the generalized Fourier expansion of elements, ℓ^2 stands in a one-to-one isomorphic and isometric correspondence with the energy spaces often constructed for the problems of elasticity. Since its properties also hold for the energy spaces, the space ℓ^2 is of great interest to those who study the boundary value problems of mechanics.

If every Cauchy sequence in a normed space has a limit in the same space, then the space is said to be complete. Thus Weierstrass's theorem states that $C(D)$ is complete.

On the same set C of functions continuous on D, let us introduce another norm:

$$\|f(x)\|_L = \int_D |f(x)|\,dx.$$

The reader can verify that this is really a norm. Is the set C with norm $\|\cdot\|_L$ complete? A single counterexample can provide a negative answer. Let $D = [0,1]$, and consider the sequence $\{g_n(x))\}$ defined by the formula

$$g_n(x) = \begin{cases} 0, & x \le 0.5, \\ n(x-0.5), & 0.5 < x \le 0.5 + 1/n, \\ 1, & x > 0.5 + 1/n. \end{cases}$$

It is clear that each $g_n(x)$ is continuous on $[0,1]$, and that $\{g_n(x)\}$ is a Cauchy sequence with respect to the norm $\|\cdot\|_L$. But the limit function $g(x)$ has a jump at $x = 0.5$, and is therefore not continuous. Thus the set C with this norm is not complete.

In the one-dimensional version of the finite element method, we meet basis functions of this type. The method also involves integration, hence the problem of sequence limits in such a space (i.e., having a norm of integral type) is not merely of academic interest.

For the sequence $\{g_n(x)\}$ the limit function is quite simple. But what happens if we try to determine the limits of all the Cauchy sequences in the set C with respect to the norm $\|\cdot\|_L$? We will encounter functions for which the Riemann integral used to define the norm does not exist. Under the construction of the integral proposed by Lebesgue, however, all the limit functions of this type become integrable. We will not discuss this further, since the interested reader can find this material in many sources on the theory of functions of a real variable. We shall only require an awareness of the fact that extremely discontinuous functions can be integrated.[5] With the Lebesgue integration we can obtain a complete space $L(D)$ with the norm $\|\cdot\|_L$.

Similarly, on the set C of continuous functions we can introduce another norm:

$$\|f(x)\|_{L^2} = \left(\int_D f^2(x)\,dx \right)^{1/2}.$$

This norm is not equivalent to the norm $\|\cdot\|_L$. The resulting normed space is not complete. A suitable augmentation of the space with limit elements (and

[5]For example, using Lebesgue integration we can state that the famous Dirichlet function

$$f(x) = \begin{cases} 1, & x \text{ rational}, \\ 0, & x \text{ irrational}, \end{cases}$$

has $\int_0^1 f(x)\,dx = 0$.

subsequent use of the Lebesgue integral, as above) gives us a complete space called $L^2(D)$. This finds wide application in mathematical physics. The inner product

$$(f, g)_{L^2} = \int_D f(x)\, g(x)\, dx$$

is easily seen to induce the norm $\|\cdot\|_{L^2}$ through the formula $\|f\| = (f, f)^{1/2}$. Hence we can say that two functions are mutually orthogonal in $L^2(D)$ if $(f, g)_{L^2} = 0$. An inner product space that happens to be complete in its induced norm is known as a Hilbert space. In $L^2(D)$ there is an orthonormal basis: that is, a set of functions $\varphi_k(x) \in L^2(D)$, $k = 1, 2, 3, \ldots$, such that

$$(\varphi_k, \varphi_n)_{L^2} = \begin{cases} 1, & i = k, \\ 0, & i \neq k. \end{cases}$$

It turns out that every function $f(x) \in L^2(D)$ can be expanded into a series

$$f(x) = \sum_{k=1}^{\infty} c_k \varphi_k(x), \qquad c_k = (f, \varphi_k)_{L^2}.$$

This is the generalized Fourier expansion of $f(x)$. The space $L^2(0, 2\pi)$ leads to the classical Fourier expansion familiar to every engineering student. The reader is aware of conditions under which such an expansion will converge uniformly to a continuous function. It can be shown that the generalized Fourier series of any function from $L^2(D)$ is convergent in the sense of the norm of this space.

Now we would like to mention another type of convergence in $L^2(D)$. It is called weak convergence. We have noted that a sequence of finite-dimensional vectors will either converge or diverge simultaneously with respect to all norms that can be imposed on the space. In particular, component-wise convergence implies convergence in this case. For an infinite-dimensional vector, the Fourier coefficients $c_k = (f, \varphi_k)_{L^2}$ play the role of the components. But component-wise convergence does not imply convergence in this case. Indeed, we note that a sequence of basis vectors $\{\varphi_k(x)\}$ is not convergent in the sense of the norm because $\|\varphi_k - \varphi_n\|_{L^2} = \sqrt{2}$ for $k \neq n$. However, for each fixed k the sequence of Fourier coefficients $c_{nk} = (\varphi_n, \varphi_k)_{L^2}$ converges to zero as $n \to \infty$. So in a certain sense the sequence $\{\varphi_n(x)\}$ does tend to zero. This so called weak convergence in the space $L^2(D)$ is not equivalent, in general, to the usual convergence with respect to the norm. Weak convergence plays an important role in modern analysis. It is often more easily established for the various limit passages that are implicated when numerical methods are applied to boundary value problems. Results pertaining to weak convergence can represent a first step toward the proof of stronger results on the convergence of numerical methods. Sometimes we can proceed no further; it depends on the complexity of the problem.

Let us mention that other Hilbert spaces are in wide use in mathematical physics, and that the results for $L^2(D)$ are easily reformulated for use with these other spaces. In particular, results on weak convergence play an important role in the justification of the finite element method for application to mechanics problems. A simple problem of this type describes a membrane in equilibrium. This can be posed with use of the minimum energy principle:

Problem. Find a function u that minimizes the energy functional

$$\frac{1}{2}\int_S \left(u_x^2 + u_y^2\right)\, dS - \int_S Fu\, dS$$

from among the functions having the fixed boundary value $u\big|_\Gamma = 0$.

Here the indices x and y denote partial derivatives with respect to the corresponding variables, and S is a bounded planar domain having sufficiently smooth boundary Γ. We must also specify the degree of smoothness possessed by the functions under comparison. Older textbooks often require continuity of all partial derivatives up to order two on the closed domain $\overline{S}$. But this same functional is used to derive the equations of the finite element method, and in that case the functions under comparison are piecewise linear. The domain S is subdivided into triangles, and on each of these a linear function is defined so that the composite function is continuous; the derivatives of the two functions defined on either side of an edge need not agree. The linear function on a triangle of the partition is completely defined by the values it takes at the nodes (i.e., the vertices of the triangle), and so all the integrals in the minimum energy principle (as well as other integrals that come into the virtual work principle for the membrane, which is a consequence of the minimum energy principle) can be represented purely in terms of the nodal values of the unknown function. Each triangle defines an area on the membrane known as a finite element. Now we see that the class of functions for comparison consists of functions lacking second derivatives on the edges of the triangles. Nonetheless, this class is quite appropriate for finding an approximate solution to the problem. The theory behind the method states that for each approximation of this type, we can find a solution. What happens when we try to consider the limit passage that results from increasing the number of finite elements on the membrane? First, we must realize that a computer cannot truly perform this limit passage. Besides the question of whether the mathematical model of the membrane is really accurate, the computer can work with only a finite number of digits: after some critical number of nodes has been reached, roundoff error will completely nullify any additional accuracy that could be provided by any further increase in the number of solution nodes. On the other hand, what if a computer could calculate with absolute precision? Here it is possible to demonstrate weak convergence for the sequence of finite-element approximations in a so called energy space where the inner product is defined by

$$(u, v)_E = \int_S \left(u_x v_x + u_y v_y\right)\, dS.$$

It is also possible to demonstrate convergence in the energy norm induced by the inner product: $\|u\|_E = (u, u)_E^{1/2}$. In both versions of convergence, the limit function lacks, in general (i.e., when we permit a non-continuous load F), continuous second derivatives. It belongs to the subspace of functions of a so called Sobolev space $W^{(1,2)}(S)$ which are equal to zero on the boundary, and on which the norm is $\|\cdot\|_E$ and all integrals are done in the Lebesgue sense. In general, the elements of $W^{(1,2)}(S)$ may completely lack the usual classical derivatives; they do possess, however, generalized derivatives that are defined through certain integral relations. In this way we come to the so called generalized (or weak) solutions of mechanics problems. It is evident that one needs some advanced background in mathematics in order to completely decipher the results of the finite element method; simple calculus will not suffice. A relative novice should learn that for elliptic problems (including problems for the membrane or a linearly elastic body), inside the domain, where the load terms have some number of continuous derivatives, the solution is smoother. The same thing occurs near the smooth portions of the boundary, and here convergence is even better than that defined by the integral norm $\|\cdot\|_E$. However, at points where the load terms are discontinuous, or at the corner points of the boundary, as a rule the energy convergence is the best we have.

Asymptotics

In engineering it is important to know not only that a sequence approximating a displacement or stress converges to the actual value, but the order of accuracy with which it converges. The latter issue falls under the heading of asymptotics. For example, we know that the series for $\sin x$, i.e.,

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots ,$$

converges for any real x; for practical purposes, however, it is important to know that when we take $|x| \leq 1$ the error involved in approximating $\sin x$ by the first n terms of the series is less than the first term omitted. This means that for practical purposes we can use partial sums of this series to calculate the values of $\sin x$ to within a prescribed bound on the error. Moreover, in practice when we realize that we can calculate, say, ten digits of the result exactly, then we need not worry too much about convergence. In this case even the error bound is of more importance from a practical viewpoint. Any engineering student is aware of the Taylor expansion of a function. The above series for $\sin x$ is such an expansion at $x_0 = 0$. The examples presented in elementary courses normally consider only the case in which the Taylor expansion is convergent and yields a series representation for a function. Although these examples are nice, they are confined to analytic functions (e.g., trigonometric or exponential functions, or analytic combinations of these). In applications, of course, we encounter many more functions that are not analytic and therefore do not possess a convergent Taylor expansion. In calculus textbooks we find theorems on remainder terms of the Taylor expansion. These theorems apply not only when the Taylor series is equal to the corresponding function, but also when we can derive only a finite number of terms of the series because some higher-order derivative of the function does not exist at $x_0 = 0$. In this case we can obtain a sum that acts as an approximation to the function, along with a bound on the error (i.e., the remainder term). We can choose to employ only some portion of this approximation, depending on our goals and desired accuracy. This is a typical example of an asymptotic expansion. Asymptotic expansions are widely used in mechanics, since they allow us to get sufficiently accurate results even when we lack exact formulas or when straightforward numerical methods fail. The latter can happen because of a singularity in the behavior of a solution or a function at some point. Such singularities can be extracted using asymptotic techniques, and the remaining regular portion can then be calculated numerically without incident.

In order to introduce asymptotic expansions, we must have the notion of an asymptotic sequence of basis functions $\varphi_k(x)$ at $x = x_0$. These functions should have the following property:

$$\varphi_{k+1}(x) = o(\varphi_k(x)) \quad \text{as } x \to x_0.$$

The notation on the right-hand side means that $\lim_{x \to x_0} \varphi_{k+1}(x)/\varphi_k(x) = 0$. In the case of Taylor's expansion we have the power functions $\varphi_n(x) = (x - x_0)^n$, where n takes on non-negative integer values. There are cases in which negative or fractional powers must appear. A sum $\sum_{k=1}^{n} c_k \varphi_k(x)$ is called an asymptotic expansion of a function $f(x)$ at $x = x_0$ of order n if, for any $k \le n$, we have

$$\lim_{x \to x_0} \frac{f(x) - \sum_{j=1}^{k} c_j \varphi_j(x)}{\varphi_k(x)} = 0.$$

This can be written in the form

$$f(x) - \sum_{j=1}^{k} c_j \varphi_j(x) = o(\varphi_k(x)).$$

Such a sum can be used to calculate the values of $f(x)$ near $x = x_0$.

An asymptotic formula can have infinite number of terms. In this case we formally write down a series that can be divergent at any point. The fact that it is an asymptotic series for $f(x)$ at $x = x_0$ is denoted by

$$f(x) \sim \sum_{j=1}^{k} c_j \varphi_j(x).$$

From an asymptotic formula it is hard to determine the neighborhood in which the error does not exceed some threshold. Sometimes, as with Taylor's expansion having the remainder in Lagrange form, one can find the bound explicitly, but in many cases it is impossible and only numerical experimentation can reveal the range of application of an asymptotic formula. Often researchers, especially those who deal with nonlinear problems, are happy to obtain even the main (i.e., first) term of an asymptotic expansion.

Asymptotics play an extremely important role in the numerical solution of those boundary value problems that involve boundary layers in their solution regions. Boundary layers can appear because of load discontinuities, breaks in the smoothness of domain boundaries, or changes in the governing equations that apply from region to region.[6] The boundary layer part of a solution can be incorporated into the finite element method. In this case — even for quite singular problems — numerical solutions can be performed with good accuracy everywhere.

For problems involving boundary layers, we typically encounter equations in which the highest-order derivative is either multiplied by a small parameter or vanishes somewhere. Consider, for example,

$$\varepsilon y'(x) + y(x) = 1, \qquad y(0) = 0.$$

Suppose the parameter ε is small and positive. For a regular Cauchy problem we could put $\varepsilon = 0$ and obtain a solution that would approximate the exact solution (this is how the leading term of an asymptotic approximation is taken for a problem exhibiting regular dependence on a parameter). In the present case, however, the attempt fails since the result is an algebraic equation $y(x) = 1$ and the initial condition cannot be satisfied. The solution to the complete problem is

$$y(x) = 1 - e^{-x/\varepsilon}.$$

The second term $e^{-x/\varepsilon}$ is very small for any fixed $x > 0$, for sufficiently small ε. This behavior is typical of the boundary layer solutions to many mechanics problems. Many specialized books are devoted to these problems, as well as to the general theory of asymptotic expansion.

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[2] Rudin, W. *Principles of Mathematical Analysis.* New York, McGraw–Hill, 1976.

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[6] One of the first examples of this type appeared in the theory of viscous liquids. Near a solid boundary, the flow of such a liquid can differ significantly from the flow elsewhere; because of this, special equations were introduced to describe the flow in a small neighborhood of the boundary. The solution in this part can change rapidly, depending on the viscosity of the liquid. The resulting flow appears as a sort of "layer" near the boundary.

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