3

Elements of Nonlinear Functional Analysis

From the viewpoint of functional analysis, nonlinear problems of mechanics are more complicated than linear problems; as in mechanics, they require new techniques for their study. Many of them, such as nonlinear elasticity in the general case, provide a wide field of investigation for mathematicians (see Antman [2]); the problem of existence of solutions in nonlinear elasticity in general is still open.

But some of the nonlinear problems of mechanics can be treated on a known background; as in the linear case, we consider only some of the known nonlinear results of functional analysis that are needed in what follows.

3.1 Fréchet and Gâteaux Derivatives

We begin nonlinear analysis of operators with definitions of differentiation. Let $F(x)$ be a nonlinear operator acting from $D(F) \subset X$ to $R(F) \subset Y$, where X and Y are real Banach spaces. Assume $D(F)$ is open.

Definition 3.1.1. $F(x)$ is *differentiable in the Fréchet sense* at $x_0 \in D(F)$ if there is a bounded linear operator, denoted by $F'(x_0)$, such that

$$F(x_0 + h) - F(x_0) = F'(x_0)h + \omega(x_0, h) \text{ for all } \|h\| < \varepsilon$$

with some $\varepsilon > 0$, where $\|\omega(x_0, h)\|/\|h\| \to 0$ as $\|h\| \to 0$. Then $F'(x_0)$ is called the *Fréchet derivative* of $F(x)$ at x_0, and $dF(x_0, h) = F'(x_0)h$ is its *Fréchet differential*. $F(x)$ is Fréchet differentiable in an open domain $S \subset D(F)$ if it is Fréchet differentiable at every point of S.

It is clear that the Fréchet derivative of a continuous linear operator is the same operator.

Problem 3.1.1. Assume $\mathbf{y} = \mathbf{f}(\mathbf{x})$ is a vector function from $\mathbb{R}^m$ to $\mathbb{R}^n$ and $\mathbf{f}(\mathbf{x}) \in (C^{(1)}(\Omega))^n$. Show that its Fréchet derivative at $\mathbf{x}_0 \in \Omega$ is the Jacobi matrix $\left(\frac{\partial f_i(\mathbf{x}_0)}{\partial x_j}\right)_{\substack{i=1,\dots,n\\ j=1,\dots,m}}$.

In the construction of the Fréchet derivative, the reader can recognize a method of the calculus of variations, used to obtain the Euler equations of a functional. The following derivative by Gâteaux is yet closer to this.

Definition 3.1.2. Assume that for all $h \in D(F)$ we have

$$\lim_{t\to 0} \frac{F(x_0+th)-F(x_0)}{t} = DF(x_0,h), \qquad x_0 \in D(F),$$

where $DF(x_0,h)$ is a linear operator with respect to h. Then $DF(x_0,h)$ is called the *Gâteaux differential* of $F(x)$ at x_0, and the operator is called *Gâteaux differentiable*. Denoting $DF(x_0,h) = F'(x_0)h$, we get the *Gâteaux derivative* $F'(x_0)$. An operator is differentiable in the Gâteaux sense in an open domain $S \subset X$ if it has a Gâteaux derivative at every point of S.

The definitions of derivatives are clearly valid for functionals. Suppose $\Phi(x)$ is a functional which is Gâteaux differentiable in a Hilbert space and that $D\Phi(x,h)$ is bounded at $x = x_0$ as a linear functional in h. Then, by the Riesz representation theorem, it can be represented in the form of an inner product; denoting the representing element by $\operatorname{grad}\Phi(x_0)$, we get

$$D\Phi(x_0,h) = (\operatorname{grad}\Phi(x_0), h).$$

By this, we have an operator $\operatorname{grad}\Phi(x_0)$ called the *gradient* of $\Phi(x)$ at x_0.

Theorem 3.1.1. If an operator $F(x)$ from X to Y is Fréchet differentiable at $x_0 \in D(F)$, then $F(x)$ is Gâteaux differentiable at x_0 and the Gâteaux derivative coincides with the Fréchet derivative.

Proof. Put th instead of h in Definition 3.1.1:

$$F(x_0+th) - F(x_0) = F'(x_0)th + \omega(x_0,th).$$

It follows that

$$\lim_{t\to 0} \frac{F(x_0+th)-F(x_0)}{t} = F'(x_0)h$$

since $\|\omega(x_0,th)\|/\|th\| \to 0$ as $t \to 0$. This means $F'(x_0)$ is a Gâteaux derivative of $F(x)$ at x_0. $\square$

Gâteaux differentiability does not imply Fréchet differentiability. We formulate a sufficient condition as

Problem 3.1.2. Assume that the Gâteaux derivative of $F(x)$ exists in a neighborhood of x_0 and is continuous at x_0 in the uniform norm of $L(X, Y)$. Show that the Fréchet derivative exists and is equal to the Gâteaux derivative.

We consider an operator equation with a parameter μ being an element of a real Banach space M:

$$F(x, \mu) = 0$$

where $D(F(x, \mu)) \subseteq X$, $R(F(x, \mu)) \subseteq Y$.

In problems of mechanics, μ can represent loads or some parameters of a body or a process (say, disturbances of the thickness of a plate or its moduli).

There are different abstract analogs of the implicit function theorem; we present two of them.

Denote by $N(x_0, r; \mu_0, \rho)$ the following neighborhood of a pair:

$$N(x_0, r; \mu_0, \rho) = \{x \in X, \mu \in M \mid \|x - x_0\| < r,\ \|\mu - \mu_0\| < \rho\}.$$

Theorem 3.1.2. Assume:

(i) $F(x_0, \mu_0) = 0$;

(ii) $F(x_0, \mu)$ is continuous with respect to μ in a ball $\|\mu - \mu_0\| < \rho_1$;

(iii) there exist $r_1 > 0$ and $\rho_1 > 0$ and a continuous linear operator A from X to Y, being continuously invertible and such that in the neighborhood $N(x_0, r_1; \mu_0, \rho_1)$

$$\|F(x, \mu) - F(y, \mu) - A(x - y)\| \le \alpha(r_1, \rho_1)\|x - y\|$$

where $\limsup_{r,\rho \to 0} |\alpha(r, \rho)|\, \|A^{-1}\| = q < 1$.

Then there exist $r_0 > 0$ and $\rho_0 > 0$ such that in $N(x_0, r_0; \mu_0, \rho_0)$ the equation

$$F(x, \mu) = 0 \tag{3.1.1}$$

has the unique solution $x = x(\mu)$ which depends continuously on μ: $x(\mu) \to x(\mu_0)$ as $\mu \to \mu_0$.

Proof. We reduce the equation to a form needed to apply the contraction mapping principle:

$$x = K(x, \mu), \qquad K(x, \mu) = x - A^{-1}F(x, \mu).$$

This equation is equivalent to (3.1.1) because A^{-1} is continuously invertible. $K(x,\mu)$ is a contraction operator with respect to x in some neighborhood of (μ_0, x_0). Indeed

$$\begin{aligned}\|K(x,\mu)-K(y,\mu)\| &= \|x-y-A^{-1}(F(x,\mu)-F(y,\mu))\|\\ &\le \|A^{-1}\|\,\|A(x-y)-(F(x,\mu)-F(y,\mu))\|\\ &\le \|A^{-1}\|\,|\alpha(r,\rho)|\,\|x-y\|\\ &\le (q+\varepsilon)\|x-y\|;\end{aligned}$$

by (iii), $q+\varepsilon<1$ if r and ρ are sufficiently small and $r<r_1$, $\rho<\rho_1$. Then there are r_0,ρ_0, $r_0\le r_1$, $\rho_0\le\rho_1$, such that $K(x,\mu)$ takes a ball $\|x-x_0\|\le r_0$ into itself when $\|\mu-\mu_0\|\le\rho_0$, indeed

$$\begin{aligned}\|K(x,\mu)-x_0\| &\le \|K(x,\mu)-K(x_0,\mu)\|+\|K(x_0,\mu)-x_0\|\\ &\le (q+\varepsilon)\|x-x_0\|+\|A^{-1}F(x_0,\mu)\|\\ &\le (q+\varepsilon)\|x-x_0\|+\|A^{-1}\|\,\|F(x_0,\mu)\|.\end{aligned}$$

Since $F(x_0,\mu)\to F(x_0,\mu_0)=0$ as $\mu\to\mu_0$, then

$$\|A^{-1}\|\,\|F(x_0,\mu)\|\le(1-q-\varepsilon)r_1 \quad \text{when} \quad \|\mu-\mu_0\|\le\rho_2 \text{ for some } \rho_2<\rho_1$$

and thus for any $r_0<r_1$, $\rho_0<\rho_2$, the ball $\|x-x_0\|\le r_0$ is taken by $K(x,\mu)$ into itself when $\|\mu-\mu_0\|\le\rho_0$.

By the contraction mapping principle, there is a solution $x=x(\mu)$ in $N(x_0,r_0;\mu_0,\rho_0)$. The continuity of $x(\mu)$ at μ_0 follows from the bound

$$\|x(\mu)-x_0\|\le\frac{\|A^{-1}\|}{1-q-\varepsilon}\|F(x_0,\mu)\|,$$

a consequence of the contraction mapping principle. □

To prove the other variant of the implicit function theorem, we need some properties of Fréchet derivatives as given by the next two lemmas.

Lemma 3.1.1. Assume an operator $F(x)$ from X to Y has a Fréchet derivative at $x=x_0$, and an operator $x=S(z)$ from a real Banach space Z to X also has a Fréchet derivative $S'(z_0)$ and $x_0=S(z_0)$. Then their composition $F(S(z))$ has a Fréchet derivative at $z=z_0$ and

$$(F(S(z_0)))'=F'(x_0)S'(z_0).$$

Proof. Substituting

$$x-x_0=S(z)-S(z_0)=S'(z_0)(z-z_0)+\omega_1(z_0,z-z_0)$$

into

$$F(x)-F(x_0)=F'(x_0)(x-x_0)+\omega(x_0,x-x_0),$$

we get

$$F(x) - F(x_0) = F'(x_0)S'(z_0)(z - z_0) + F'(x_0)\omega_1(z_0, z - z_0) + \\ + \omega(x_0, S(z) - S(z_0)).$$

This completes the proof, since the last two terms on the right-hand side are of the order $o(\|z - z_0\|)$. □

The next lemma is the so-called *Lagrange identity.*

Lemma 3.1.2. Assume that $F(x)$ from X to Y is Fréchet differentiable in a neighborhood Ω of x_0. Then for $x \in \Omega$ we have

$$F(x) - F(x_0) = \int_0^1 F'(x_0 + \theta(x - x_0))\, d\theta\, (x - x_0).$$

Proof. By Lemma 3.1.1, the composition $F(S(\theta))$, where $S(\theta) = x_0 + \theta(x - x_0)$, has a Fréchet derivative

$$\frac{d}{d\theta} F(S(\theta)) = F'(x_0 + \theta(x - x_0))(x - x_0)$$

since $S'(\theta) = x - x_0$. Integrating this over $[0, 1]$ with regard for continuity of $F(S(\theta))$ in θ, we complete the proof. □

We can now present the more traditional version of the implicit function theorem. In preparation we introduce a partial Fréchet derivative $F_x(x, \mu)$ of $F(x, \mu)$ with respect to x as its Fréchet derivative with respect to x when μ is fixed.

Theorem 3.1.3. Assume:

(i) $F(x_0, \mu_0) = 0$;

(ii) for some $r > 0$ and $\rho > 0$, the operator $F(x, \mu)$ is continuous on the set $N(x_0, r; \mu_0, \rho)$;

(iii) $F_x(x, \mu)$ is continuous at (x_0, μ_0);

(iv) $F_x(x_0, \mu_0)$ has a continuous inverse linear operator.

Then there exist $r_0 > 0$, $\rho_0 > 0$ such that the equation $F(x, \mu) = 0$ has the unique solution $x = x(\mu)$ in a ball $\|x - x_0\| \le r_0$ when $\|\mu - \mu_0\| \le \rho_0$. If there is, in addition, $F_\mu(x, \mu)$ which is continuous at (x_0, μ_0) then $x(\mu)$ has a Fréchet derivative at $\mu = \mu_0$ and

$$x'(\mu_0) = -F_x^{-1}(x_0, \mu_0)F_\mu(x_0, \mu_0).$$

Proof. We verify that $A = F_x(x_0, \mu_0)$ meets condition (iii) of Theorem 3.1.2. Consider

$$\Psi(x, y, \mu) = \|F(x, \mu) - F(y, \mu) - F_x(x, \mu_0)(x - y)\|.$$

By Lemma 3.1.2,

$$F(x, \mu) - F(y, \mu) = \int_0^1 F_x(y + \theta(x - y), \mu)\, d\theta\, (x - y)$$

and so

$$\begin{aligned}\Psi(x, y, \mu) &= \left\| \int_0^1 (F_x(y + \theta(x - y), \mu) - F_x(x_0, \mu_0))\, d\,\theta(x - y) \right\| \\ &\le \int_0^1 \|F_x(y + \theta(x - y), \mu) - F_x(x_0, \mu_0)\|\, d\theta\, \|x - y\| \\ &\le \alpha(r, \rho)\|x - y\|\end{aligned}$$

where

$$\alpha(r, \rho) = \sup_{x,\mu} \|F_x(x, \mu) - F_x(x_0, \mu_0)\| \quad \text{ on } N(x_0, r; \mu_0, \rho)$$

is such that $\alpha(r, \rho) \to 0$ as $r, \rho \to 0$ since $F_x(x, \mu)$ is continuous at (x_0, μ_0). The other conditions of Theorem 3.1.2 are also satisfied and so a solution $x = x(\mu)$ actually exists. We leave the second part of the theorem on differentiability of $x(\mu)$ without proof. □

Using the implicit function theorem, we can determine whether a solution to a problem depends continuously and uniquely on some parameters.

We studied several linear problems of mechanics with constant parameters. The reader can now verify that small disturbances of elastic moduli or, say, the thickness of a plate, bring small disturbances in displacements (small in a corresponding energy norm). We note that for linear problems this can be shown more easily by using the contraction mapping principle, but in nonlinear problems using the implicit function theorem is more convenient.

3.2 Liapunov–Schmidt Method

We shall say that (x_0, μ_0) is a *regular point* of the equation $F(x, \mu) = 0$ if there is a neighborhood of (x_0, μ_0), say $N(x_0, r; \mu_0, \rho)$, in which there is a unique solution $x = x(\mu)$.

The implicit function theorem gives sufficient conditions for regularity of $F(x, \mu)$ at (x_0, μ_0).

In mechanics, the breakdown of the property of regularity of a solution is of great importance; it is usually connected with some qualitative change of the properties of a system under consideration: its behavior, stability, or type of motion.

We now consider an important class of non-regular points of an operator equation.

Definition 3.2.1. (x_0, μ_0) is a *bifurcation point* of the equation $F(x, \mu) = 0$ if for any $r > 0$, $\rho > 0$, in the ball $\|\mu - \mu_0\| \le \rho$ there exists μ such that in the ball $\|x - x_0\| \le r$ there are at least two solutions of the equation corresponding to μ.

Many problems of mechanics (in particular, in shell theory) are such that in an energy space a partial Fréchet derivative $F_x(x_0, \mu_0)$ of a corresponding operator of a problem may be reduced to the form $I - B$, $B = B(x_0, \mu_0)$, where B is a compact linear operator (as a rule it is self-adjoint) and so the results of the Fredholm–Riesz–Schauder theory are valid. In particular, $I - B$ is not continuously invertible if and only if there is a nontrivial solution to $(I - B)x = 0$, and this is the case when the implicit function theorem is not applicable. This case is now considered.

Without loss of generality, we assume $x_0 = 0$, $\mu_0 = 0$ (we can always change $x \mapsto x_0 + x$, $\mu \mapsto \mu_0 + \mu$) so let

$$F(0, 0) = 0.$$

Suppose F is an operator acting from $H \times M$ in H where H is a Hilbert space and M is a real Banach space. As we said, we suppose that $F_x(0, 0)$ takes the form

$$F_x(0, 0) = I - B_0$$

with B_0 a compact self-adjoint linear operator in H.

The equation $F(x, \mu) = 0$ can be rewritten in the form

$$(I - B_0)x = -F(x, \mu) + (I - B_0)x$$

or

$$(I - B_0)x = R(x, \mu), \qquad R(x, \mu) = -F(x, \mu) + (I - B_0)x. \tag{3.2.1}$$

We now consider the Liapunov–Schmidt method of determining the dependence of solution to (3.2.1) on μ when $\|\mu\|$ is small and there are nontrivial solutions to the equation $(I - B_0)x = 0$. As in Section 2.11, denote by N the set of these nontrivial solutions and let $x_1, \ldots, x_n$ be an orthonormal basis of N.

In the beginning of the proof of Theorem 2.11.4 we saw that the operator

$$Q_0 x = (I - B_0)x + \sum_{k=1}^{n} (x, x_k) x_k$$

is continuously invertible. Equation (3.2.1) can be written in the form

$$Q_0 x = R(x,\mu) + \sum_{k=1}^{n} \alpha_k x_k, \qquad \alpha_k = (x, x_k). \tag{3.2.2}$$

We now consider (3.2.2) as an equation with respect to x that has parameters $\mu, \alpha_1, \dots, \alpha_n$, introducing, in preparation,

$$x = u + \sum_{k=1}^{n} \beta_k x_k, \qquad (u, x_j) = 0, \quad j = 1, \dots, n.$$

Here $u \in M$, M being the orthogonal complement of N in H. As $(x, x_k) = \alpha_k$, then $x = u + \sum_{k=1}^{n} \alpha_k x_k$ and (3.2.2) is

$$Q_0 u = R\left(u + \sum_{k=1}^{n} \alpha_k x_k, \mu\right). \tag{3.2.3}$$

This equation defines u as a function of the variables $\mu, \alpha_1, \dots, \alpha_n$. Since $R_x(0,0) = -F_x(0,0) + (I - B_0) = 0$ we get

$$\left(Q_0 x - R(u + \sum_{k=1}^{n} \alpha_k x_k, \mu)\right)_u \Bigg|_{\substack{u=0 \\ \mu=0, \alpha_1 = \cdots = \alpha_n = 0}} = Q_0$$

where Q_0 is a continuously invertible operator, so all the conditions of the implicit function theorem are fulfilled. Therefore (3.2.3) has a unique solution for every μ, $\alpha_1, \dots, \alpha_n$ when $\|\mu\|$ and $|\alpha_k|$ are small:

$$u = u(\mu, \alpha_1, \dots, \alpha_n).$$

This solution must be orthogonal to all x_k, $k = 1, \dots, n$, and to define values $\alpha_1, \dots, \alpha_n$ we have the system

$$(u(\mu, \alpha_1, \dots, \alpha_n), x_k) = 0, \qquad k = 1, \dots, n \tag{3.2.4}$$

which is called the Liapunov–Schmidt equation of branching.

Using the Liapunov–Schmidt method one can investigate so-called post-critical behavior of a system, say, post-buckling of a von Kármán plate.

3.3 Critical Points of a Functional

From now on, we shall consider operators and real-valued functionals given in a real Hilbert space H. So let $\Phi(x)$ be a functional on H.

Definition 3.3.1. $x_0 \in H$ is called a *local minimal* (*maximal*) point of $\Phi(x)$ if there is a ball $B = \{x \mid \|x - x_0\| \le \varepsilon\}$, $\varepsilon > 0$, such that for all $x \in B$ we have $\Phi(x) \ge \Phi(x_0)$ ($\Phi(x) \le \Phi(x_0)$). Minimal and maximal points are called *extreme points* of $\Phi(x)$. If $\Phi(x) \ge \Phi(x_0)$ for all $x \in H$, then x_0 is a point of *absolute minimum* of $\Phi(x)$.

We prove the following

Theorem 3.3.1. Assume:

(i) $\Phi(x)$ is given on an open set $S \subset H$;

(ii) there exists $\operatorname{grad}\Phi(x)$ at $x = x_0 \in S$;

(iii) x_0 is an extreme point of $\Phi(x)$.

Then $\operatorname{grad}\Phi(x_0) = 0$.

Proof. Let h be an arbitrary element of H. The functional $\Phi(x_0 + th)$ is a function in a real variable t that attains its minimum at $t = 0$. Since

$$\left.\frac{d\Phi(x_0 + th)}{dt}\right|_{t=0} = 0,$$

we have

$$(\operatorname{grad}\Phi(x_0), h) = 0. \tag{3.3.1}$$

Since h is arbitrary, the conclusion follows. □

Definition 3.3.2. A point x_0 at which $\operatorname{grad}\Phi(x_0) = 0$ is called a *critical point* of $\Phi(x)$.

In fact, we implicitly used this theorem for linear problems when $\Phi(x)$ was a (quadratic) functional of total energy of an elastic body and (3.3.1) was an equation defining a generalized solution of the corresponding problem. Similar results will be valid for some nonlinear problems in what follows.

In preparation, we introduce some definitions.

Definition 3.3.3. A functional $\Phi(x)$ is called *weakly continuous* at $x = x_0$ if for every sequence $\{x_k\}$ converging weakly to x_0 the numerical sequence $\Phi(x_k)$ tends to $\Phi(x_0)$ as $k \to \infty$. It is called weakly continuous on an open set $S \subset H$ if it is weakly continuous at every point of S.

Definition 3.3.4. A functional $\Phi(x)$ given on H is called *growing* if

$$\inf_{\|x\|=R} \Phi(x) \to \infty \text{ as } R \to \infty.$$

We obtained a necessary condition for existence of critical points of a functional. Now we point out some sufficient conditions for this that have important applications in mechanics.

Lemma 3.3.1. Assume Q is a weakly closed and bounded set in H. A weakly continuous functional $\Phi(x)$ is bounded on Q and attains its minimal and maximal values in it.

Proof. First we prove that the values of $\Phi(x)$ on Q are bounded from above. If not, there is a sequence $\{x_n\} \subset Q$ such that $\Phi(x_n) \to \infty$ as $n \to \infty$. By hypothesis $\{x_n\}$ contains a subsequence $\{x_{n_k}\}$ weakly convergent to $x_0 \in Q$ and so

$$\Phi(x_{n_k}) \to \Phi(x_0) \neq \infty \text{ as } n_k \to \infty,$$

which contradicts the assumption. Boundedness from below is thus clearly seen.

Let $d = \inf_{x \in Q} \Phi(x)$. By definition of infimum there is a sequence $\{z_n\}$ for which $\Phi(z_n) \to d$ as $n \to \infty$. As above, it contains a subsequence $\{z_{n_k}\}$ converging weakly to $z_0 \in Q$. By weak continuity of $\Phi(x)$ we get $\Phi(z_0) = d$. The proof for the maximal value is similar. □

Note that a ball $B(R) = \{x \mid \|x\| \leq R\}$ has the properties of Q of the lemma.

In what follows, some problems of mechanics can be reduced to a problem of finding critical points of the functional

$$\Psi(x) = \|x\|^2 + \Phi(x)$$

with $\Phi(x)$ a weakly continuous functional. The functional $\Psi(x)$ is not weakly continuous because of the term $\|x\|^2$ and so Lemma 3.3.1 does not apply.

Theorem 3.3.2. Let $\Phi(x)$ be a weakly continuous functional. On a ball $B(R) = \{x \mid \|x\| \leq R\}$, the functional $\Psi(x) = \|x\|^2 + \Phi(x)$ attains its minimal value.

Proof. By Lemma 3.3.1, $\Phi(x)$ and hence $\Psi(x)$ is bounded from below on $B(R)$. Let $d = \inf \Psi(x)$ on $B(R)$ and $\{x_n\}$ be a sequence in $B(R)$ such that $\Psi(x_n) \to d$ as $n \to \infty$. By weak compactness of $B(R)$ we can produce a subsequence $\{x_{n_k}\}$ which converges weakly to $x_0 \in B(R)$. Moreover, from the bounded numerical sequence $\{\|x_{n_k}\|\}$ we can take a subsequence which tends to some number a, $a \leq R$. Redenote the last subsequence as $\{x_n\}$ again.

We show that $\|x_0\| \leq a$. Indeed, since $x_n \rightharpoonup x_0$ then $\lim_{n\to\infty}(x_n, x_0) = \|x_0\|^2$ and we have

$$\|x_0\|^2 = \lim_{n\to\infty} |(x_n, x_0)| \leq \lim_{n\to\infty} \|x_n\| \, \|x_0\| = a\|x_0\|$$

which gives $\|x_0\| \leq a$.

By weak continuity of $\Phi(x)$, we get $\Phi(x_n) \to \Phi(x_0)$ as $n \to \infty$ and $\Psi(x_n) \to d = a^2 + \Phi(x_0)$ simultaneously. Since $x_0 \in B(R)$,

$$\Psi(x_0) = \|x_0\|^2 + \Phi(x_0) \geq \inf_{x \in B(R)} \Psi(x) = d = a^2 + \Phi(x_0),$$

and so $\|x_0\| \geq a$. With the above, this implies $\|x_0\| = a$ and thus x_0 is a point at which $\Psi(x)$ takes its minimal value on $B(R)$. □

Remark 3.3.1. Since $\{x_{n_k}\}$ from the proof converges weakly to x_0 and the sequence $\{\|x_{n_k}\|\}$ converges to $\|x_0\| = a$, this sequence converges to x_0 strongly in H.

Definition 3.3.5. Assume $\inf \Phi(x) = d > -\infty$ on H. A sequence $\{x_n\}$ is called a *minimizing sequence* of $\Phi(x)$ if $\Phi(x_n) \to d$ as $n \to \infty$.

In the proof of Theorem 3.3.2 we have established that under the conditions of that theorem any sequence minimizing $\Psi(x)$ contains a subsequence that converges strongly to an element at which the minimum of $\Psi(x)$ occurs. Now we can formulate

Theorem 3.3.3. Assume that a functional $\Psi(x) = \|x\|^2 + \Phi(x)$, where $\Phi(x)$ is weakly continuous on H, is growing. Then:

(i) there exists $x_0 \in H$ at which $\Psi(x)$ takes its minimal value;

(ii) any minimizing sequence of $\Psi(x)$ contains a subsequence which converges strongly to a point at which $\Psi(x)$ takes its minimal value: moreover, every weakly convergent subsequence of $\{x_n\}$ converges strongly to a minimizer of $\Psi(x)$;

(iii) if a point x_0 at which $\Psi(x)$ takes its minimal value is unique, then a minimizing sequence converges to x_0 strongly;

(iv) if $\operatorname{grad} \Phi(x_0)$ exists at a point of minimum x_0, then

$$2x_0 + \operatorname{grad} \Phi(x_0) = 0.$$

Proof. By Theorem 3.3.2, on a ball $\|x\| \leq R$ the functional $\Psi(x)$ takes its minimal value. Since $\Psi(x)$ is growing we can take R so large that the minimum is attained inside the open ball $\|x\| < R$. So statements (i) and (ii) follow from Theorem 3.3.3 and Remark 3.3.1. Statement (iv) follows from Theorem 3.3.1. The proof of (iii) is carried out in a way similar to that given in Section 1.23. □

Now we consider the application of the Ritz method to solve the problem of minimizing $\Psi(x)$ under the restrictions of Theorem 3.3.3. First we state the equations of Ritz's method. Let $g_1, g_2, g_3, \ldots$ be a complete system in H such that every finite subsystem is linearly independent. Denote by H_n a subspace of H which is spanned by $g_1, \ldots, g_n$.

The approximation of the Ritz method to minimize the functional $\Psi(x)$ is now formulated as follows:

- Find a minimizer x_n of $\Psi(x)$ on H_n.

- Note that if $\Psi(x)$ has $\operatorname{grad}\Psi(x)$ then the equations to find the nth Ritz approximation are

$$(\operatorname{grad}\Psi(x_n), g_k) = 0, \qquad k = 1, \ldots, n, \quad x_n \in H_n.$$

Theorem 3.3.4. Under the restrictions of Theorem 3.3.3, the following hold:

(i) for each n there exists a solution $x_n \in H_n$, the nth Ritz approximation of the minimizer of $\Psi(x)$;

(ii) the sequence of Ritz approximations is a minimizing sequence of $\Psi(x)$, and thus

(iii) the sequence $\{x_n\}$ contains at least one weakly convergent subsequence whose weak limit is a minimizer of $\Psi(x)$ — in fact, this subsequence converges strongly to the minimizer;

(iv) every weakly convergent subsequence of $\{x_n\}$ converges strongly to a minimizer of $\Psi(x)$; if a minimizer of $\Psi(x)$ is unique, then the whole sequence $\{x_n\}$ converges to it strongly.

Proof. (i) Solvability of the problem for the nth approximation of solution by the Ritz method follows from Theorem 3.3.3.

(ii) Let x_0 be a solution to the main problem

$$\Psi(x_0) = d = \inf_{x \in H} \Psi(x).$$

As the system $g_1, g_2, g_3, \ldots$ is complete, there is $x^{(n)} \in H_n$ such that

$$\|x_0 - x^{(n)}\| = \delta_n \to 0 \text{ as } n \to \infty.$$

Since $\Psi(x)$ is continuous we get

$$|\Psi(x^{(n)}) - \Psi(x_0)| = \varepsilon_n \to 0 \text{ as } n \to \infty.$$

But x_n is a minimizer of $\Psi(x)$ on H_n, so

$$d = \Psi(x_0) \le \Psi(x_n) = \inf_{x \in H_n} \Psi(x) \le \Psi(x^{(n)}).$$

Therefore

$$|\Psi(x_n) - \Psi(x_0)| \le \varepsilon_n \to 0 \text{ as } n \to \infty$$

and thus $\{x_n\}$ is a minimizing sequence of $\Psi(x)$.

The other statements follow from Theorem 3.3.3. $\square$

Note that Theorem 3.3.4 can be applied to linear and nonlinear problems of mechanics.

3.4 Von Kármán Equations of a Plate

Theorem 3.3.4 will be applied to the boundary value problem of equilibrium of a plate described by the von Kármán equations, which are

$$\Delta^2 w = [f, w] + q \quad \text{in } \Omega \subset \mathbb{R}^2, \tag{3.4.1}$$

$$\Delta^2 f = -[w, w] \quad \text{in } \Omega, \tag{3.4.2}$$

where $w(x, y)$ is the normal displacement of the middle surface Ω of the plate, $f(x, y)$ is the Airy function, $q = q(x, y)$ is the transverse external load, and

$$[u, v] = \frac{\partial^2 u}{\partial x^2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\frac{\partial^2 v}{\partial y^2} - 2\frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 v}{\partial x \partial y}.$$

We consider the Dirichlet problem for these equations:

$$w\big|_{\partial\Omega} = \frac{\partial w}{\partial n}\Big|_{\partial\Omega} = 0, \tag{3.4.3}$$

$$f\big|_{\partial\Omega} = \frac{\partial f}{\partial n}\Big|_{\partial\Omega} = 0. \tag{3.4.4}$$

Let us consider the integro-differential equations

$$a(w, \varphi) = B(f, w, \varphi) + \int_\Omega q\varphi \, d\Omega, \tag{3.4.5}$$

$$a(f, \eta) = -B(w, w, \eta), \tag{3.4.6}$$

where

$$a(w, \varphi) = \int_\Omega \left\{ \frac{\partial^2 w}{\partial x^2}\left(\frac{\partial^2 \varphi}{\partial x^2} + \nu\frac{\partial^2 \varphi}{\partial y^2}\right) + 2(1-\nu)\frac{\partial^2 w}{\partial x \partial y}\frac{\partial^2 \varphi}{\partial x \partial y} + \right.$$
$$\left. + \frac{\partial^2 w}{\partial y^2}\left(\frac{\partial^2 \varphi}{\partial y^2} + \nu\frac{\partial^2 \varphi}{\partial x^2}\right)\right\} d\Omega,$$

$$B(f, w, \varphi) = \int_\Omega \left\{ \left(\frac{\partial^2 f}{\partial x \partial y}\frac{\partial w}{\partial y} - \frac{\partial^2 f}{\partial y^2}\frac{\partial w}{\partial x}\right)\frac{\partial \varphi}{\partial x} + \right.$$
$$\left. + \left(\frac{\partial^2 f}{\partial x \partial y}\frac{\partial w}{\partial x} - \frac{\partial^2 f}{\partial x^2}\frac{\partial w}{\partial y}\right)\frac{\partial \varphi}{\partial y}\right\} d\Omega,$$

ν being Poisson's ratio, $0 < \nu < 1/2$.

Note that $a(u, v)$ is the scalar product (1.10.4) (with an omitted multiplier — the bending rigidity) of the energy space E_{PC} for an isotropic plate, and we shall use this notation in this section.

Suppose that (3.4.5) and (3.4.6), with respect to the unknown function w, f, being smooth (of $C^{(4)}(\overline{\Omega})$) and satisfying the boundary conditions (3.4.3) and (3.4.4), are valid for every φ, η which also satisfy (3.4.3) for these functions and their normal derivatives on the boundary. The usual tools of the calculus of variations show that the pair (w, f) is a classical solution to the von Kármán equations (3.4.1) and (3.4.2). This means that we can use (3.4.5) and (3.4.6) to define a generalized solution to the problem under consideration. We note that (3.4.5) expresses the virtual work principle for the plate, and (3.4.6) is the equation of compatibility. So we introduce

Definition 3.4.1. A pair (w, f), $w, f \in E_{PC}$, is called a generalized solution to the problem (3.4.1)–(3.4.4) if it satisfies the integro-differential equations (3.4.5)–(3.4.6) for any (φ, η), $\varphi, \eta \in E_{PC}$.

For correctness of the definition the load $q = q(x, y)$ must be such that the term $\int_\Omega q\varphi \, d\Omega$ is a continuous linear functional in E_{PC}; for this it suffices that, say, q be of $L^1(\Omega)$ (cf., Section 1.14).

Under the restrictions of the definition, all terms in (3.4.5) and (3.4.6) make sense as each of the first derivatives of any of the functions under consideration are of $L^p(\Omega)$ with any $p < \infty$. Indeed, a typical term which is not present in a linear statement of the plate problem is bounded as

$$\left| \int_\Omega \frac{\partial^2 f}{\partial x^2} \frac{\partial w}{\partial y} \frac{\partial \varphi}{\partial y} \, d\Omega \right| \leq \left(\int_\Omega \left| \frac{\partial^2 f}{\partial x^2} \right|^2 \, d\Omega \right)^{1/2} \cdot \\ \cdot \left(\int_\Omega \left| \frac{\partial w}{\partial y} \right|^4 \, d\Omega \right)^{1/4} \left(\int_\Omega \left| \frac{\partial \varphi}{\partial y} \right|^4 \, d\Omega \right)^{1/4}, \qquad (3.4.7)$$

and hence is finite.

We could present a functional whose gradient in the space $E_{PC} \times E_{PC}$ is defined by (3.4.5) and (3.4.6); unfortunately it is not of the form required by Theorem 3.3.4. That is why we shall reformulate the problem with respect to the only unknown function w, defining f as an operator with respect to w and construct a functional of w whose critical point is a generalized solution of the problem. We now embark on this program.

So let w be a fixed but arbitrary element of E_{PC}. Consider $B(w, w, \eta)$ as a functional with respect to η in E_{PC}. It is clearly linear. By (3.4.7) written for a typical term with $f = w$, thanks to the imbedding theorem in E_{PC}, we get

$$|B(w, w, \eta)| \leq m \|w\|_{E_P}^2 \|\eta\|_{E_P},$$

i.e., the functional is continuous and so we can apply the Riesz representation theorem to get

$$-B(w, w, \eta) = (c, \eta)_{E_P} = a(c, \eta).$$

Being uniquely defined by $w \in E_{PC}$, the element $c \in E_{PC}$ can be considered as a value of a nonlinear operator

$$c = C(w), \qquad a(C(w), \eta) = -B(w, w, \eta). \qquad (3.4.8)$$

Before studying the properties of C we introduce

Definition 3.4.2. An operator A mapping from a Banach space X to a Banach space Y is called *compact* if it is continuous in X and takes every bounded set of X into a precompact set in Y. An operator is called *completely continuous* if it takes every weakly convergent sequence of X, $x_n \rightharpoonup x_0$, into a sequence $A(x_n)$ converging strongly to $A(x_0)$.

Lemma 3.4.1. A completely continuous operator F mapping a Hilbert space X into a Banach space Y is compact.

Proof. F is continuous since when a sequence $\{x_n\}$ converges to x_0 strongly in X then it converges to x_0 weakly, too.

Next we take a bounded set S in X and let $\{x_n\}$ be a sequence lying in S. From $\{x_n\}$, thanks to its boundedness, we can choose a subsequence $\{x_{n_k}\}$ converging weakly to $x_0 \in X$. Then, by definition of complete continuity, we get the sequence $\{F(x_{n_k})\}$ converging to $F(x_0)$ strongly. This means $F(S)$ is precompact, hence F is compact. $\square$

It is known that there are compact operators in a Hilbert space which are not completely continuous.

Corollary 3.4.1. If $F(x)$ is a completely continuous operator, then the functional $\|F(x)\|^2$ is a weakly continuous functional in X.

The proof is evident. Now we can prove

Lemma 3.4.2. The operator $C(w)$ defined by (3.4.8) is completely continuous.

Proof. When the functions $u, v, w \in E_{PC}$ are smooth, direct integration by parts gives

$$B(u, v, w) = B(v, u, w) = B(v, w, u) = B(w, u, v); \tag{3.4.9}$$

the limit passage shows that this is valid for $u, v, w \in E_{PC}$. So

$$-B(w, w, \eta) = \int_\Omega \left\{ \left(\frac{\partial w}{\partial x}\right)^2 \frac{\partial^2 \eta}{\partial y^2} + \left(\frac{\partial w}{\partial y}\right)^2 \frac{\partial^2 \eta}{\partial x^2} - 2\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\frac{\partial^2 \eta}{\partial x \partial y} \right\} d\Omega.$$

Next we take an arbitrary sequence $\{w_n\}$ converging weakly to w_0 in E_{PC} and consider

$$|a(C(w_n) - C(w_0), \eta)| = |B(w_n, w_n, \eta) - B(w_0, w_0, \eta)|.$$

Using the Hölder inequality, we bound a typical term of the right-hand side of this equality as follows:

$$\begin{aligned} d_n &= \left| \int_\Omega \left[\left(\frac{\partial w_n}{\partial x} \right)^2 - \left(\frac{\partial w_0}{\partial x} \right)^2 \right] \frac{\partial^2 \eta}{\partial y^2} \, d\Omega \right| \\ &= \left| \int_\Omega \left(\frac{\partial w_n}{\partial x} - \frac{\partial w_0}{\partial x} \right) \left(\frac{\partial w_n}{\partial x} + \frac{\partial w_0}{\partial x} \right) \frac{\partial^2 \eta}{\partial y^2} \, d\Omega \right| \\ &\le \left\| \frac{\partial w_n}{\partial x} - \frac{\partial w_0}{\partial x} \right\|_{L^4(\Omega)} \left(\left\| \frac{\partial w_n}{\partial x} \right\|_{L^4(\Omega)} + \left\| \frac{\partial w_0}{\partial x} \right\|_{L^4(\Omega)} \right) \left\| \frac{\partial^2 \eta}{\partial y^2} \right\|_{L^2(\Omega)} . \end{aligned}$$

By the imbedding theorem in E_{PC}, which is a subspace of $W^{2,2}(\Omega)$, we get

$$d_n \le m_1 \left\| \frac{\partial w_n}{\partial x} - \frac{\partial w_0}{\partial x} \right\|_{L^4(\Omega)} (\|w_n\|_{E_P} + \|w_0\|_{E_P}) \|\eta\|_{E_P}$$

and, thanks to the boundedness of a weakly convergent sequence,

$$d_n \le m_2 \|w_n - w_0\|_{W^{1,4}(\Omega)} \|\eta\|_{E_P}$$

where m_1 and m_2 are constants.

Gathering all such bounds, we obtain

$$|a(C(w_n) - C(w_0), \eta)| \le m_3 \|w_n - w_0\|_{W^{1,4}(\Omega)} \|\eta\|_{E_P} .$$

Putting $\eta = C(w_n) - C(w_0)$, we finally obtain

$$\|C(w_n) - C(w_0)\|_{E_P} \le m_3 \|w_n - w_0\|_{W^{1,4}(\Omega)} \to 0 \text{ as } n \to \infty$$

since the imbedding operator of $W^{2,2}(\Omega)$ into $W^{1,4}(\Omega)$ is completely continuous (a particular case of Sobolev's imbedding theorems in $W^{2,2}(\Omega)$). The last limit passage shows that C is completely continuous. □

From this lemma we see that (3.4.6) with a given $w \in E_{PC}$ has the unique solution

$$f = C(w). \tag{3.4.10}$$

If $\{w_n\}$ converges to w_0 weakly in E_{PC}, then $\{f_n\} = \{C(w_n)\}$ converges to $f_0 = C(w_0)$ strongly in E_{PC}.

From now on we consider f in (3.4.5) to be determined by (3.4.10).

For a fixed $w \in E_{PC}$, by bounds of the type (3.4.7), we see that the functional

$$B(f, w, \varphi) + \int_\Omega q\varphi \, d\Omega$$

is linear and continuous with respect to $\varphi \in E_{PC}$. So applying the Riesz representation theorem, we have a representation

$$B(f, w, \varphi) + \int_\Omega q\varphi \, d\Omega = a(U, \varphi)$$

where $U \in E_{PC}$ is uniquely determined by $w \in E_{PC}$; so we define an operator G, $U = G(w)$, acting in E_{PC}, by

$$B(f, w, \varphi) + \int_\Omega q\varphi \, d\Omega = a(G(w), \varphi). \tag{3.4.11}$$

In much the same way that Lemma 3.4.2 is proved we can establish

Lemma 3.4.3. G is a completely continuous operator in E_{PC}.

Now the following is evident:

Lemma 3.4.4. The system of equations (3.4.5)–(3.4.6) defining a generalized solution of the problem under consideration is equivalent to the operator equation

$$w = G(w) \tag{3.4.12}$$

with a completely continuous operator G acting in E_{PC}.

Now we introduce a functional

$$I(w) = \frac{1}{2}a(w, w) + \frac{1}{4}a(f, f) - \int_\Omega qw \, d\Omega$$

where, as we said, f is defined by (3.4.8).

The decisive point of this section is

Lemma 3.4.5. For every $w \in E_{PC}$, we have

$$\operatorname{grad} I(w) = w - G(w). \tag{3.4.13}$$

Proof. In accordance with the definition of the gradient of a functional, we consider

$$\frac{dI(w + t\varphi)}{dt}\Big|_{t=0} = \frac{1}{2}\frac{d}{dt}a(w + t\varphi, w + t\varphi)\Big|_{t=0} + \frac{1}{2}a\left(f, \frac{df}{dt}\right)\Big|_{t=0} - \int_\Omega q\varphi \, d\Omega$$

where $f = C(w + t\varphi)$. It is clear that

$$\frac{1}{2}\frac{d}{dt}a(w + t\varphi, w + t\varphi)\Big|_{t=0} = a(w, \varphi).$$

Using the definition (3.4.8) of C, with regard for the equality $B(w, \varphi, \eta) = B(\varphi, w, \eta)$, a particular case of (3.4.9), we calculate directly that

$$a\left(\frac{df}{dt}\Big|_{t=0}, \eta\right) = -2B(w, \varphi, \eta)$$

and so

$$a\left(f, \frac{df}{dt}\right)\Big|_{t=0} = -2B(w, \varphi, f) = -2B(f, w, \varphi).$$

It follows that

$$\left.\frac{dI(w+t\varphi)}{dt}\right|_{t=0} = a(w,\varphi) - B(f,w,\varphi) - \int_\Omega q\varphi\, d\Omega$$

and, thanks to (3.4.11),

$$\left.\frac{dI(w+t\varphi)}{dt}\right|_{t=0} = a(w,\varphi) - a(G(w),\varphi) = a(w - G(w),\varphi).$$

This, by definition of the gradient of a functional, means that (3.4.13) holds. □

Combining Lemmas 3.4.3 and 3.4.4, we have

Lemma 3.4.6. A critical point w of $I(w)$ defines the pair $(w, G(w))$ that is a generalized solution of the problem under consideration.

So we reduce the problem of finding a generalized solution of the problem to the problem of the minimum of a functional (it is not equivalent as there are in general solutions which are not points of minimum of the functional).

To apply Theorem 3.3.3, it remains to verify

Lemma 3.4.7. The functional $2I(w)$ is growing and has the form

$$\|w\|^2_{E_P} + \Phi_1(w)$$

where

$$\Phi_1(w) = \frac{1}{2}a(f,f) - 2\int_\Omega qw\, d\Omega$$

is a weakly continuous functional, f being defined by (3.4.10).

Proof. $2I(w)$ is growing since

$$2I(w) \geq a(w,w) - 2\left|\int_\Omega qw\, d\Omega\right| = \|w\|^2_{E_P} - 2\left|\int_\Omega qw\, d\Omega\right|$$

and

$$2I(w) \geq \|w\|^2_{E_P} - m\|w\|_{E_P} \to \infty \quad \text{if } \|w\|_{E_P} \to \infty$$

as q is assumed to be such that $\int_\Omega qw\, d\Omega$ is a continuous functional with respect to $w \in E_{PC}$.

Weak continuity of $\Phi_1(w)$ is a consequence of Corollary 3.4.1 and Lemma 3.4.2 for $a(f,f) = \|C(w)\|^2_{E_P}$ and the fact that the continuous linear functional $\int_\Omega qw\, d\Omega$ is weakly continuous (by definition). □

So we can reformulate Theorem 3.3.3 in the case of the plate problem as follows

Theorem 3.4.1. Assume q is such that $\int_\Omega qw\,d\Omega$ is a continuous linear functional with respect to w in E_{PC}. Then any critical point of the growing functional $I(w)$ which has at least one point of absolute minimum is a generalized solution of the plate problem in the sense of Definition 3.4.1; any minimizing sequence of $I(w)$ contains at least one subsequence which converges strongly to a generalized solution of the problem; each of the weak limit points of the minimizing sequence, which are strong limit points simultaneously, is a generalized solution to the problem under consideration.

The reader can also reformulate Theorem 3.3.4 in the present case to justify application of the Ritz method (and thus the method of finite elements) to von Kármán equations. Note that in this modification of the method we must find f exactly from (3.4.6). But it is not too difficult to show that f can be found approximately, also by the Ritz method, and the corresponding theorem on convergence remains valid in the present case.

3.5 Buckling of a Thin Elastic Shell

Following an article by I.I. Vorovich [27] (and [28]), we now consider a buckling problem for a shallow elastic shell described by equations of von Kármán's type. We want to study stability of the momentless state (here $w = 0$) of the shell. Assume the external load to be proportional to a parameter λ. For every λ, existence of the momentless state of the shell is seen. We formulate the equations of equilibrium as follows:

$$\begin{aligned}\Delta^2 w &= -\lambda\left(T_1\frac{\partial^2 w}{\partial x^2} + T_2\frac{\partial^2 w}{\partial y^2} + 2T_{12}\frac{\partial^2 w}{\partial x\partial y} - F_1\frac{\partial w}{\partial x} - F_2\frac{\partial w}{\partial y}\right) + \\ &\quad + [f, w+z], \\ \Delta^2 f &= -\left\{2[z,w] + [w,w]\right\}. \end{aligned} \tag{3.5.1}$$

We study a problem with Dirichlet conditions

$$w\big|_{\partial\Omega} = \frac{\partial w}{\partial n}\Big|_{\partial\Omega} = f\big|_{\partial\Omega} = \frac{\partial f}{\partial n}\Big|_{\partial\Omega} = 0. \tag{3.5.2}$$

Here $z = z(x,y) \in C^{(3)}(\overline{\Omega})$ is the equation of mid-surface of the shell. It is supposed that the tangential stresses T_1, T_2, T_{12} are given, belong to $L^2(\Omega)$ and, as assumed during derivation of the equations, satisfy equations of the two-dimensional theory of elasticity with forces (F_1, F_2). Other bits of notation are taken from the previous section.

The equations of a generalized statement of the problem under consideration are as follows:

$$a(w,\varphi) = \lambda \int_\Omega \left[T_1 \frac{\partial w}{\partial x}\frac{\partial \varphi}{\partial x} + T_2 \frac{\partial w}{\partial y}\frac{\partial \varphi}{\partial y} + T_{12}\left(\frac{\partial w}{\partial x}\frac{\partial \varphi}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \varphi}{\partial x}\right)\right] dx\,dy$$
$$+ B(f, w+z, \varphi), \tag{3.5.3}$$

$$a(f,\eta) = -2B(z,w,\eta) - B(w,w,\eta). \tag{3.5.4}$$

Using standard variational tools, we can derive from these the equations (3.5.1) if a solution is assumed to be sufficiently smooth; conversely, we can derive (3.5.1) from (3.5.3)–(3.5.4). So we can take the latter equations to formulate

Definition 3.5.1. A pair w, f from E_{PC} is called a generalized solution to the problem (3.5.1)–(3.5.2) if it satisfies the integro-differential equations (3.5.3)–(3.5.4) for any $\varphi, \eta \in E_{PC}$

The problem under consideration has a trivial solution $w = f = 0$. We are interested in when there exists a nontrivial solution, i.e., in solving a nonlinear eigenvalue problem.

First we mention that, as in Section 3.4, we solve the equation (3.5.4) and then exclude $f \in E_{PC}$ from the equation (3.5.3) using the solution f of (3.5.4) when $w \in E_{PC}$ is given. It is clear that

$$f = f_1 + f_2$$

where the f_i are defined by the equations

$$a(f_1,\eta) = -2B(z,w,\eta), \qquad a(f_2,\eta) = -B(w,w,\eta).$$

Using the Riesz representation theorem we can find from these that

$$f_1 = Lw, \qquad f_2 = C(w). \tag{3.5.5}$$

In Section 3.4 it was shown that $C(w)$ is a completely continuous operator. The same is valid for the linear operator L (we leave it to the reader to show this).

In Section 2.5, we introduced the self-adjoint bounded operator C that is now redenoted as K. It is defined by

$$a(Kw,\varphi) = \int_\Omega \left[T_1 \frac{\partial w}{\partial x}\frac{\partial \varphi}{\partial x} + T_{12}\left(\frac{\partial w}{\partial x}\frac{\partial \varphi}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \varphi}{\partial x}\right) + T_2 \frac{\partial w}{\partial y}\frac{\partial \varphi}{\partial y}\right] dx\,dy.$$

K is compact in E_{PC} as follows from Sobolev's imbedding theorem.

Applying the Riesz representation theorem to the relation (3.5.3) wherein f is defined by (3.5.5), we find an operator equation for a generalized solution of the problem under consideration

$$w - G(\lambda, w) = 0. \tag{3.5.6}$$

The next point is to define a functional whose critical points are solutions to (3.5.6). It is

$$I(\lambda, w) = \frac{1}{2}a(w,w) + \frac{1}{4}a(f,f) - \lambda J(w)$$

where

$$J(w) = \frac{1}{2}\int_\Omega \left[T_1 \left(\frac{\partial w}{\partial x}\right)^2 + 2T_{12}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} + T_2\left(\frac{\partial w}{\partial y}\right)^2 \right] dx\, dy.$$

$I(\lambda, w)$ is the total energy of the system "shell-load."

Lemma 3.5.1. For every $w \in E_{PC}$ we have

$$\operatorname{grad} I(\lambda, w) = w - G(\lambda, w). \tag{3.5.7}$$

The proof is similar to that for Lemma 3.4.4 and is omitted, as is the proof that $G(\lambda, w)$ is a completely continuous operator in $w \in E_{PC}$.

Next we consider the functional $a(f,f)$. It is seen that

$$\begin{aligned} a(f,f) &= a(f_1,f_1) + A_3(w) + A_4(w), \\ A_3(w) &= 2a(f_1,f_2) = -4B(z,w,f_2), \\ A_4(w) &= a(f_2,f_2) = \frac{1}{2}B(f_2,w,w). \end{aligned}$$

Here $A_k(w)$ is a homogeneous function of order k with respect to w, i.e.,

$$A_k(tw) = t^k A_k(w).$$

We leave it to the reader to show that $a(f,f)$, along with each of its parts, is a weakly continuous functional on E_{PC} (for $a(f,f)$, this is a consequence of Corollary 3.4.1).

It is evident that $J(w)$ is a weakly continuous functional in E_{PC}. So we have

Lemma 3.5.2. For every real number λ, the functional $I(\lambda, w)$ takes the form

$$I(\lambda, w) = \frac{1}{2}\|w\|^2_{E_{PC}} + \Psi(\lambda, w), \qquad \Psi(\lambda, w) = \frac{1}{4}a(f,f) - \lambda J(w),$$

where $\Psi(\lambda, w)$ is a weakly continuous functional.

From now on, we assume that

$$J(w) > 0 \quad \text{if } w \neq 0,\ w \in E_{PC}. \tag{3.5.8}$$

This assumption has the physical implication that almost everywhere in the shell the stress state of the shell is compressive.

To study stability of the non-buckled state of the shell (that is, when $w = 0$), beginning from L. Euler's work on stability of a bar, one solves the linearized (here around zero state) eigenvalue problem that is now

$$\operatorname{grad}\left[\frac{1}{2}a(w,w) + \frac{1}{4}a(f_1,f_1)\right] = \lambda \operatorname{grad} J(w). \tag{3.5.9}$$

The lowest eigenvalue of the latter, denoted λ_E and called the Euler lowest critical value, is usually considered as a value when the main, trivial form of equilibrium of the shell becomes unstable. We shall analyze this method for the shell.

We begin with the eigenvalue problem (3.5.9).

Lemma 3.5.3. There is a countable set λ_k of eigenvalues $\lambda_k > 0$ of the equation (3.5.9) considered in E_{PC}.

Proof. We first mention that the scalar product

$$\langle w, \varphi \rangle = a(w,\varphi) + \frac{1}{2}a(Lw, L\varphi), \qquad f_1 = Lw,$$

induces the norm in E_{PC} which is equivalent to the usual one since

$$a(w,w) \le \langle w, w\rangle \le m\, a(w,w).$$

Using the new norm, we can rewrite (3.5.9) in the form

$$w = \lambda K_1 w$$

where K_1 is determined, thanks to the Riesz representation theorem, by the equality

$$\langle K_1 w, \varphi\rangle = \int_\Omega \left[T_1 \frac{\partial w}{\partial x}\frac{\partial \varphi}{\partial x} + T_{12}\left(\frac{\partial w}{\partial x}\frac{\partial \varphi}{\partial y} + \frac{\partial w}{\partial y}\frac{\partial \varphi}{\partial x}\right) + T_2 \frac{\partial w}{\partial y}\frac{\partial \varphi}{\partial y}\right] dx\, dy.$$

It is easily seen that K_1, as well as K, is strictly positive, self-adjoint, and compact, and thus we can use Theorem 2.14.2 which gives even more than the lemma states. $\square$

For the trivial solution $w = f = 0$, the total energy $I(\lambda, w) = 0$. A state of the shell at which $I(\lambda, w)$ takes its minimal value is, in a certain sense, stable. So it is of interest what is the range of λ in which $I(\lambda, w)$ can take negative values.

Theorem 3.5.1. Assume $T_1, T_{12}, T_2 \in L^2(\Omega)$ and w_E is an eigenfunction of the linearized boundary value problem (3.5.9) corresponding to its smallest eigenvalue λ_E, the Euler critical value. Then for every λ of the half-line

$$\lambda > \lambda^* \equiv \lambda_E - \frac{A_3^2(w_E)}{4A_4(w_E)J(w_E)} \tag{3.5.10}$$

there exists at least one nontrivial solution of the nonlinear boundary value problem (3.5.6) at which $I(\lambda, w)$ is negative.

The proof is a consequence of the following three lemmas. The first of them is auxiliary.

Lemma 3.5.4. Assume that $w \in E_{PC}$ satisfies

$$\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 = 0 \tag{3.5.11}$$

in the sense of $L^1(\Omega)$ (almost everywhere in Ω). Then $w = 0$.

Proof. If $w \in C^{(2)}(\Omega)$, then (3.5.11) means the Gaussian curvature of the surface $z = w(x, y)$ vanishes so the surface is developable and, thanks to the boundary conditions (3.5.2), $w = 0$.

If $w \notin C^{(2)}(\Omega)$, we take another route. For arbitrary $w \in E_{PC}$, $F \in W^{2,2}(\Omega)$, the following formula holds:

$$\begin{aligned}\int_\Omega \left[\left(\frac{\partial^2 F}{\partial x \partial y}\frac{\partial w}{\partial y} - \frac{\partial^2 F}{\partial y^2}\frac{\partial w}{\partial x}\right)\frac{\partial w}{\partial x} + \left(\frac{\partial^2 F}{\partial x \partial y}\frac{\partial w}{\partial x} - \frac{\partial^2 F}{\partial x^2}\frac{\partial w}{\partial y}\right)\frac{\partial w}{\partial y}\right] dx\, dy \\ = 2\int_\Omega \left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right] F\, dx\, dy.\end{aligned} \tag{3.5.12}$$

(This is easily seen after integrating by parts for smooth functions; the limit passage shows that it is valid for the needed classes.) In (3.5.12) we put

$$F = \frac{1}{2}(x^2 + y^2)$$

which gives for w satisfying (3.5.11)

$$\int_\Omega \left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] dx\, dy = 0.$$

This, together with the boundary conditions for w, completes the proof. □

Lemma 3.5.5. The functional $I(\lambda, w)$ is growing for every $\lambda > 0$; that is, we have $I(\lambda, w) \to \infty$ as $\|w\|_{E_P} \to \infty$.

Proof. On the unit sphere $S = \{w\colon a(w, w) = 1\}$ of E_{PC} consider the set S_1 defined by

$$\frac{1}{2}a(w, w) - \lambda J(w) > \frac{1}{4}.$$

Then on the image of S_1 under the mapping $w \mapsto Rw$, we get

$$\begin{aligned}I(\lambda, Rw) &\ge \frac{1}{2}a(Rw, Rw) - \lambda J(Rw) \\ &= R^2\left[\frac{1}{2}a(w, w) - \lambda J(w)\right] \\ &> \frac{1}{4}R^2, \qquad w \in S_1.\end{aligned} \tag{3.5.13}$$

Next consider $I(\lambda, Rw)$ when $w \in S_2 = S \setminus S_1$. Here

$$\frac{1}{2}a(w,w) - \lambda J(w) \le \frac{1}{4}. \tag{3.5.14}$$

Let us introduce the weak closure of S_2 in E_{PC}, denoted by $\operatorname{Cl} S_2$. First we show that $\operatorname{Cl} S_2$ does not contain zero. If to the contrary it does contain zero then there is a sequence $\{w_n\} \in \operatorname{Cl} S_2$ such that $a(w_n, w_n) = 1$ and $w_n \rightharpoonup 0$ in E_{PC} (or, equivalently, in $W^{2,2}(\Omega)$). By the imbedding theorem in $W^{2,2}(\Omega)$, the sequences of first derivatives of $\{w_n\}$ tend to zero strongly in $L^p(\Omega)$ for any $p < \infty$ and thus $J(w_n) \to 0$, which contradicts (3.5.14) since

$$\frac{1}{2} \equiv \frac{1}{2}a(w_n, w_n) \le \frac{1}{4} + \lambda J(w_n).$$

Next we show that for all $w \in \operatorname{Cl} S_2$,

$$A_4(w) \ge c_* \tag{3.5.15}$$

wherein c_* is a positive constant. Indeed, if (3.5.15) is not valid there is a sequence $\{w_n\} \in \operatorname{Cl} S_2$ such that $A_4(w_n) \to 0$ as $n \to \infty$. This sequence contains a subsequence which converges weakly to w_0 belonging to $\operatorname{Cl} S_2$ too. Since A_4 is a weakly continuous functional,

$$A_4(w_0) = 0.$$

This means that

$$a(f_2, f_2) = 0, \qquad f_2 = C(w_0).$$

Returning to (3.5.5), we get

$$B(w_0, w_0, \eta) = 0$$

or, equivalently,

$$\int_\Omega \left[\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2} - \left(\frac{\partial^2 w_0}{\partial x \partial y}\right)^2 \right] \eta \, dx \, dy = 0$$

for any $\eta \in E_{PC}$. As E_{PC} is dense in $L^2(\Omega)$,

$$\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2} - \left(\frac{\partial^2 w_0}{\partial x \partial y}\right)^2 = 0$$

almost everywhere in Ω and, by Lemma 3.5.3, it follows that $w_0(x, y) = 0$. This contradicts the fact that w_0 belongs to $\operatorname{Cl} S_2$ which does not contain zero.

Since $|A_3(w)| \le c_1$ on S, we get, thanks to (3.5.15),

$$I(\lambda, Rw) \ge c_* R^4 - \left(\frac{1}{4}R^2 + c_1 R^3\right)$$

when $w \in \mathrm{Cl}\, S_2$ and so for sufficiently large R, with regard for (3.5.13), we obtain

$$I(\lambda, Rw) \geq \frac{1}{4} R^2$$

for all $w \in S$. This means that $I(\lambda, w)$ is growing. □

By Theorem 3.3.3 it follows that, for any λ, the functional $I(\lambda, w)$ takes its minimal value in E_{PC}. But $w = 0$ is also a critical point of the functional, so to conclude the proof of Theorem 3.5.1 we formulate

Lemma 3.5.6. Under the conditions of Theorem 3.5.1, the minimal value of $I(\lambda, w)$ is negative if λ satisfies (3.5.10).

Proof. Consider $I(\lambda, cw_E)$ where c is a constant. It is seen that

$$I(\lambda, cw_E) = c^2 \left[\frac{1}{2} a(w_E, w_E) + \frac{1}{4} a(Lw_E, Lw_E) - \lambda J(w_E) \right] + \\ + c^3 A_3(w_E) + c^4 A_4(w_E), \qquad (f_1 = Lw_E).$$

Further, from (3.5.9) it follows that

$$\frac{1}{2} a(w_E, w_E) + \frac{1}{4} a(Lw_E, Lw_E) = \lambda_E J(w_E).$$

Hence

$$I(\lambda, cw_E) = c^2 \left[(\lambda_E - \lambda) J(w_E) + c A_3(w_E) + c^2 A_4(w_E) \right].$$

The minimum of $I(\lambda, cw_E)/c^2$ considered as a function of the real variable c is taken at

$$c_0 = -\frac{1}{2} A_3(w_E)/A_4(w_E);$$

this minimum is equal to

$$\min_c (c^{-2} I(\lambda, cw_E)) = (\lambda_E - \lambda) J(w_E) - A_3^2(w_E)/A_4(w_E).$$

So for λ satisfying (3.5.10), we get

$$I(\lambda, c_0 w_E) < 0$$

and thus at w_0, a minimizer of $I(\lambda, w)$ at the same λ,

$$I(\lambda, w_0) < 0.$$

This completes the proof of the lemma, and therefore of Theorem 3.5.1. □

A very important result follows from Theorem 3.5.1.

Corollary 3.5.1. Assume that there is an eigenfunction w_E corresponding to the Euler critical value λ_E such that

$$A_3(w_E) \neq 0.$$

In this case we have a sharp inequality $\lambda^* < \lambda_E$.

This result is of fundamental importance in the theory of stability of shells, since from it we have that if $A_3(w_E) \neq 0$, then the problem of stability cannot be solved by linearization in the neighborhood of a momentless state of stress, used since Euler in the theory of stability of rods. If $A_3(w_E) \neq 0$, then we must investigate the problem of stability of a shell in its nonlinear formulation.

Theorem 3.5.2. Let $T_1, T_2, T_{12} \in L^2(\Omega)$. Then there is a value $\lambda_l \leq \lambda^*$ such that for any $\lambda < \lambda_l$ the nonlinear problem (3.5.6) has the unique solution $w = 0$.

Proof. Assume w is a solution of (3.5.6), i.e., the pair $w, f = Lw + C(w)$ from E_{PC} satisfies (3.5.3)–(3.5.4) for arbitrary $\varphi, \eta \in E_{PC}$. Setting $\varphi = w$ and $\eta = f$ in (3.5.3)–(3.5.4) we get

$$\begin{aligned} a(w, w) &= 2\lambda J(w) + B(f, w, w) + B(f, z, w), \\ a(f, f) &= -2B(z, w, f) - B(w, w, f). \end{aligned}$$

Summing these equalities term by term, we have the identity

$$a(w, w) + a(f, f) = 2\lambda J(w) - B(z, f, w). \tag{3.5.16}$$

Using the elementary inequality $|ab| \leq a^2 + \frac{1}{4}b^2$, we get an estimate

$$\begin{aligned} |B(z, f, w)| &= \left| \int_\Omega \left(\frac{\partial^2 f}{\partial x^2} \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \frac{\partial^2 z}{\partial y^2} - 2 \frac{\partial^2 f}{\partial x \partial y} \frac{\partial^2 z}{\partial x \partial y} \right) w \, dx \, dy \right| \\ &\leq \int_\Omega \left[\left(\frac{\partial^2 f}{\partial x^2} \right)^2 + \left(\frac{\partial^2 f}{\partial y^2} \right)^2 + 2 \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2 \right] dx \, dy + \\ &\quad + \frac{1}{4} \int_\Omega \left[\left(\frac{\partial^2 z}{\partial x^2} \right)^2 + \left(\frac{\partial^2 z}{\partial y^2} \right)^2 + 2 \left(\frac{\partial^2 z}{\partial x \partial y} \right)^2 \right] w^2 \, dx \, dy. \end{aligned}$$

Integrating by parts in the expression for $a(f, f)$ gives

$$a(f, f) = \int_\Omega \left[\left(\frac{\partial^2 f}{\partial x^2} \right)^2 + \left(\frac{\partial^2 f}{\partial y^2} \right)^2 + 2 \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2 \right] dx \, dy$$

and thus, from (3.5.16), it follows that

$$a(w, w) \leq 2\lambda J(w) + \frac{1}{4} \int_\Omega \left[\left(\frac{\partial^2 z}{\partial x^2} \right)^2 + \left(\frac{\partial^2 z}{\partial y^2} \right)^2 + 2 \left(\frac{\partial^2 z}{\partial x \partial y} \right)^2 \right] w^2 \, dx \, dy. \tag{3.5.17}$$

Now we need a lemma which will be proved later.

Lemma 3.5.7. On the surface $S = \{w \mid J(w) = 1\}$ in E_{PC}, the functional

$$I_1(w) = a(w,w) - \frac{1}{4}\int_\Omega \left[\left(\frac{\partial^2 z}{\partial x^2}\right)^2 + \left(\frac{\partial^2 z}{\partial y^2}\right)^2 + 2\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2\right] w^2\, dx\, dy$$

has finite minimum denoted by $2\lambda^{**}$.

We are continuing the proof. From this lemma, it follows that

$$I_1(w) \geq 2\lambda^{**} J(w)$$

since all of the functionals are homogeneous with respect to w of order 2. Thus, from (3.5.17), we get

$$(2\lambda^{**} - 2\lambda) J(w) \leq 0,$$

from which it follows that if $\lambda \leq \lambda^{**}$ then

$$J(w) \leq 0.$$

This is possible only at $w = 0$, and the proof is complete. □

Proof of Lemma 3.5.7. Assume $\{w_n\}$ is a minimizing sequence of $I_1(w)$ on S and, by contradiction, that the minimum on S is not finite, i.e., $I_1(w_n) \to -\infty$ as $n \to \infty$. It is quite obvious that $\|w_n\|_{E_P} \to \infty$.

Define $w_n^* = w_n/\|w_n\|_{E_P}$. We can consider the sequence $\{w_n^*\}$ to be weakly convergent to an element $w_0^* \in E_{PC}$. In this case

$$J(w_n) = \|w_n\|^2_{E_P} J(w_n^*)$$

so

$$J(w_n^*) = J(w_n)/\|w_n\|^2_{E_P} \to 0 \text{ as } n \to \infty.$$

Since J is weakly continuous then $J(w_0^*) = 0$ and thus $w_0^* = 0$. This means that $w_n^* \rightharpoonup 0$.

By the imbedding theorem we get

$$\sup_\Omega |w_n^*(x,y)| \to 0$$

and so

$$a_n = \int_\Omega \left[\left(\frac{\partial^2 z}{\partial x^2}\right)^2 + \left(\frac{\partial^2 z}{\partial y^2}\right)^2 + 2\left(\frac{\partial^2 z}{\partial x \partial y}\right)^2\right] (w_n^*)^2\, dx\, dy \to 0 \text{ as } n \to \infty.$$

Thus

$$\lim_{n\to\infty} I_1(w_n) = \lim_{n\to\infty} \|w_n\|^2_{E_P}\left(1 - \frac{1}{4}a_n\right) = +\infty,$$

a contradiction. Similar considerations demonstrate that a minimizing sequence $\{w_n\}$ of I_1 is bounded. Then there is a subsequence that converges weakly to an element w_0. This element belongs to S since $J(w)$ is weakly continuous. The structure of I_1 provides that $I_1(w_0) = \lambda^{**}$. □

As a result of Theorem 3.5.2 we get the estimates

$$-\infty < \lambda^{**} \leq \lambda_l \leq \lambda^* \leq \lambda_E < \infty. \tag{3.5.18}$$

From the statement of Lemma 3.5.7, it is seen that λ^{**} can be defined as the lowest eigenvalue of the boundary value problem

$$\operatorname{grad} I_1(w) = 2\lambda \operatorname{grad} J(w). \tag{3.5.19}$$

Let us consider a particular case of a von Kármán plate. Here $z(x, y) = 0$ and thus the problem (3.5.19) takes the form

$$\operatorname{grad}(a(w, w)) = 2\lambda \operatorname{grad} J(w).$$

But the equation (3.5.9) determining the λ_E for the plate coincides with this one as $f_1 = Lw = 0$ for a plate. Thus $\lambda_E = \lambda^{**}$ and (3.5.18) states that $\lambda_l = \lambda_E$ for the plate. This implies an important

Theorem 3.5.3. In the case of a plate ($z(x, y) = 0$), under the conditions of Theorem 3.5.1, the equality $\lambda_l = \lambda_E$ is satisfied. In other words, for $\lambda \leq \lambda_E$ there is a unique generalized solution, $w = 0$, of the problem under consideration; if $\lambda > \lambda_E$ then there is another solution of the problem, at which the functional of total energy of the plate is strictly negative.

This theorem establishes the possibility of applying Euler's method of linearization to the problem of stability of a plate.

We note that many works (not mentioned here) are devoted to mathematical questions in the theory of von Kármán's plates and shells. The corresponding boundary value problems of the theory are a touchstone of abstract nonlinear mathematical theory because of their importance in applications, as well as their not too complicated form.

3.6 The Nonlinear Problem of Equilibrium of the Theory of Elastic Shallow Shells

We consider another simple modification of the nonlinear theory of elastic shallow shells when the geometry of the mid-surface of the shell is identified with the geometry of a plane. This modification of the theory is widely applied in engineering calculations. Nonlinear theory of shallow shells in curvilinear coordinates is considered in [26] in detail.

We express the equations describing the behavior of the shell in a notation which is commonly used along with this version of the theory. Namely, we let x, y denote the coordinates on the plane that is identified with the mid-surface of the shell, u, v denote the tangential components of the vector of displacements of the mid-surface, w denote the transverse displacement of the mid-surface, and subscripts x, y denote partial derivatives with respect to x and y. The equations of equilibrium of the shell are

$$D\nabla^4 w + N_1(k_1 - w_{xx}) + N_2(k_2 - w_{yy}) - 2N_{12}w_{xy} - F = 0, \tag{3.6.1}$$

$$\begin{aligned}
&\nabla^2 u + (1+\mu)/(1-\mu)(u_x + v_y)_x + \\
&\quad + 2/(1-\mu)[(k_1 w)_x + w_x w_{xx} + \mu(k_2 w)_x + \mu w_y w_{xy}] + \\
&\quad + w_y w_{xy} + w_x w_{yy} = 0, \\
&\nabla^2 v + (1+\mu)/(1-\mu)(u_x + v_y)_y + \\
&\quad + 2/(1-\mu)[(k_2 w)_y + w_y w_{yy} + \mu(k_1 w)_y + \mu w_x w_{xy}] + \\
&\quad + w_x w_{xy} + w_y w_{xx} = 0,
\end{aligned} \tag{3.6.2}$$

D, E, μ being the elastic constants, $0 < \mu < 1/2$. We consider the shell under the action of a transverse load F. The components of the tangential strain tensor are

$$\varepsilon_1 = u_x + k_1 w + \frac{1}{2}w_x^2, \quad \varepsilon_2 = v_y + k_2 w + \frac{1}{2}w_y^2, \quad \varepsilon_{12} = u_y + v_x + w_x w_y. \tag{3.6.3}$$

Let us formulate the conditions under which we justify application of Ritz's method to a boundary value problem for the shell, and so for the finite element method as well, and establish an existence theorem.

We suppose Ω, the domain occupied by the shell, satisfies the same conditions we imposed earlier for the von Kármán plate. Let the shell be clamped against the transverse translation at three points (x_i, y_i), $i = 1, 2, 3$, that do not lie on the same straight line:

$$w(x_i, y_i) = 0. \tag{3.6.4}$$

It is sufficient (but not necessary) to assume that

$$w\big|_{\Gamma_1} = 0 \tag{3.6.5}$$

holds on a portion Γ_1 of the boundary.

Let us call C_4 the set of functions w belonging to $C^{(4)}(\Omega)$ and satisfying the conditions (3.6.4)–(3.6.5).

For the tangential displacements u, v, the minimal restrictions in this consideration must be such that Korn's inequality of two-dimensional elasticity holds. That is (see Mikhlin [19]), we must have

$$\int_\Omega (u^2 + v^2 + u_x^2 + u_y^2 + v_x^2 + v_y^2)\,dx\,dy \le m\int_\Omega [u_x^2 + (u_y + v_x)^2 + v_y^2]\,dx\,dy. \tag{3.6.6}$$

One of the possible restrictions under which (3.6.6) holds for all u, v with the unique constant m is

$$u\big|_{\Gamma_2} = 0, \qquad\qquad v\big|_{\Gamma_2} = 0, \tag{3.6.7}$$

Γ_2 being some part of the boundary of Ω.

Let us introduce the set C_2 of vector functions $\omega = (u, v)$ with the components belonging to $C^{(2)}(\Omega)$ and satisfying (3.6.7).

We may suppose that some part of the boundary of the shell is elastically supported (the corresponding term of the energy of the system should be included into the expression of the energy norm) or that on some part of the boundary there is given a transverse load (here the term that is the work of the load on the boundary must be included into the energy functional). We will not place these conditions in the differential form; they are well known and can be derived from the variational statement of the problem. The presence of these conditions has no practical impact on the way in which we consider the problem.

Let us introduce energy spaces. Let E_1 be a subspace of $W^{1,2}(\Omega) \times W^{1,2}(\Omega)$ that is the completion of the set C_2 in the norm of $W^{1,2}(\Omega) \times W^{1,2}(\Omega)$. The Korn inequality (3.6.6) implies that on E_1 the following norm is equivalent:

$$\|\omega\|_{E_1}^2 = \frac{Eh}{2(1-\mu^2)} \int_\Omega [e_1^2 + e_2^2 + 2\mu e_1 e_2 + \frac{1}{2}(1-\mu)e_{12}^2]\, dx\, dy,$$

where

$$e_1 = u_x, \qquad e_2 = v_y, \qquad e_{12} = u_y + v_x,$$

and h is the shell thickness.

E_2, a subspace of $W^{2,2}(\Omega)$, is the completion of C_4 in the norm of $W^{2,2}(\Omega)$. On E_2 there is an equivalent norm (the energy norm we introduced for the problem of bending of the plate):

$$\|w\|_{E_2}^2 = \frac{1}{2} D \int_\Omega [(\nabla^2 w)^2 + 2(1-\mu)(w_{xy}^2 - w_{xx} w_{yy})]\, dx\, dy.$$

The norms on E_i induce the inner products that are denoted with use of the names of corresponding spaces. Denote $E_1 \times E_2$ by E.

Definition 3.6.1. $\mathbf{u} = (u, v, w) \in E$ is called a generalized solution of the problem of equilibrium of a shallow shell if for an arbitrary $\delta\mathbf{u} = (\delta u, \delta v, \delta w) \in E$ it satisfies the equation

$$\begin{aligned} \int_\Omega (M_1 \delta\kappa_1 + M_2 \delta\kappa_2 + 2M_{12}\delta\chi + N_1 \delta\epsilon_1 + N_2 \delta\epsilon_2 + N_{12}\delta\epsilon_{12})\, dx\, dy \\ = \int_\Omega F \delta w\, dx\, dy + \int_{\partial\Omega} f \delta w\, ds, \end{aligned} \tag{3.6.8}$$

where

$$M_1 = D(\kappa_1 + \mu\kappa_2), \qquad M_2 = D(\kappa_2 + \mu\kappa_1), \qquad M_{12} = D(1-\mu)\chi,$$

$$N_1 = \frac{Eh}{1-\mu^2}(\epsilon_1 + \mu\epsilon_2), \quad N_2 = \frac{Eh}{1-\mu^2}(\epsilon_2 + \mu\epsilon_1), \quad N_{12} = \frac{Eh}{2(1+\mu)}\epsilon_{12},$$

$$\kappa_1 = -w_{xx}, \qquad \kappa_2 = -w_{yy}, \qquad \chi = -w_{xy},$$

f being the external load on the edge of the shell.

We note that on the part of the boundary where $\delta w = 0$, it is not necessary to show f. However we shall assume that on this part of the boundary the function $f = 0$.

It is seen that all the stationary points of the energy functional

$$I(\mathbf{u}) = \|w\|_{E_2}^2 + \frac{1}{2}\int_\Omega (N_1\epsilon_1 + N_2\epsilon_2 + N_{12}\epsilon_{12})\,dx\,dy - \\ - \int_\Omega Fw\,dx\,dy - \int_{\partial\Omega} fw\,ds \tag{3.6.9}$$

are solutions to (3.6.8) since moving all the terms of (3.6.8) to the left-hand side we get on the left in (3.6.8) the expression for the first variation of the functional $I(\mathbf{u})$.

Let us note that for the correctness of Definition 3.6.1 it is necessary to impose an additional requirement: the terms

$$\int_\Omega F\delta w\,dx\,dy + \int_{\partial\Omega} f\delta w\,ds$$

must make sense for any $\delta w \in E_2$. The set of these loads is called E^*. By Sobolev's imbedding theorems, sufficient conditions for the loads to belong to E^* are:

$$F = F_0 + F_1$$

where $F_0 \in L(\Omega)$ and F_1 is a finite sum of δ-functions (point transverse forces);

$$f = f_0 + f_1$$

where $f_0 \in L(\partial\Omega)$ and f_1 is a finite sum of δ-functions (point transverse forces on $\partial\Omega$). Under these conditions, the functional

$$\int_\Omega F\delta w\,dx\,dy + \int_{\partial\Omega} f\delta w\,ds$$

is linear and continuous in $\delta w \in E_2$.

By the Riesz representation theorem there exists the unique element $g \in E_2$ such that

$$\int_\Omega F\delta w\,dx\,dy + \int_{\partial\Omega} f\delta w\,ds = (g, \delta w)_{E_2}. \tag{3.6.10}$$

Now we can represent $I(\mathbf{u})$ in a more compact form:

$$I(\mathbf{u}) = \|w\|_{E_2}^2 + \frac{1}{2}\int_\Omega (N_1\epsilon_1 + N_2\epsilon_2 + N_{12}\epsilon_{12})\,dx\,dy - (g, w)_{E_2}. \quad (3.6.11)$$

Let us find the tangential displacements u_1, u_2 through w. For this consider the equation

$$\int_\Omega (N_1\delta\epsilon_1 + N_2\delta\epsilon_2 + N_{12}\delta\epsilon_{12})\,dx\,dy = 0$$

in E_1. Reasoning as was done earlier, we can easily establish that this equation is uniquely solvable in E_1 with respect to $\omega = (u_1, u_2)$; the solution can be written as

$$\omega = G(w),$$

where G is a completely continuous operator. Let us put this ω into the expression of $I(\mathbf{u})$. After this substitution, the functional $I(\mathbf{u})$ depends only on w; it is denoted by $\aleph(w)$. Standard reasoning leads us to the statement that any stationary point of $\aleph(w)$ is a generalized solution of the problem under consideration.

The functional $\aleph(w)$ has a structure that is suitable for application of Theorem 3.3.4. To justify the Ritz method it is enough to show that $\aleph(w)$ is growing. Let us demonstrate this.

Lemma 3.6.1. Let the external load belong to E^*. Then $\aleph(w)$ is growing; that is, $\aleph(w) \to \infty$ when $\|w\|_{E_2} \to \infty$.

Proof. The proof follows from considering the form of $\aleph(w)$. Indeed, under the above assumptions, we have

$$N_1\epsilon_1 + N_2\epsilon_2 + 2N_{12}\epsilon_{12} \geq 0.$$

Then

$$|(g, \delta w)_{E_2}| \leq \|g\|_{E_2}\|w\|_{E_2},$$

so

$$\aleph(w) \geq \|w\|_{E_2}^2 - \|g\|_{E_2}\|w\|_{E_2}.$$

From this the lemma follows. □

Thus we have

Theorem 3.6.1. Let the conditions of Lemma 3.6.1 hold. Then

(i) there is a generalized solution of the problem of equilibrium of the shell that belongs to E_2 and admits a minimum of the functional $\aleph(w)$;

(ii) any sequence $\{w_n\}$ minimizing the functional $\aleph(w)$ in E_2 contains a subsequence that converges strongly to a generalized solution of the problem;

(iii) the equations of the Ritz method (and thus of Galerkin's method and so of any conforming modification of the finite element method) have a solution in each approximation; the set of approximations contains a subsequence that converges strongly to a generalized solution of the problem in E_2; moreover, any weakly converging subsequence of the Ritz approximations converges strongly to a generalized solution of the problem.

3.7 Degree Theory

This is only a sketch of degree theory of a map, which will be used in what follows. We begin with an illuminating example.

Let $f(z)$ be a function holomorphic on a closed domain D of the complex plane, and let ∂D, the boundary of D, be smooth and let it not contain zeros of $f(z)$. Then, as is well known, the number defined by the integral

$$n = \oint_{\partial D} \frac{f'(z)}{f(z)}\, dz$$

is equal to the number of zeros of $f(z)$ inside D with regard for their multiplicity.

This is extended to more general classes of maps; this is the so-called degree theory, a full presentation of which can be found in Schwartz [21].

The degree of a finite-dimensional vector-field $\mathbf{\Phi}(\mathbf{x})\colon \mathbb{R}^n \to \mathbb{R}^n$, originally due to L.E.J. Brouwer, is defined as follows. Let $\mathbf{\Phi}(\mathbf{x}) = (\Phi_1(\mathbf{x}), \ldots, \Phi_n(\mathbf{x}))$ be continuously differentiable on a bounded open domain D with the boundary ∂D in $\mathbb{R}^n$. Suppose $\mathbf{p} \in \mathbb{R}^n$ does not belong to ∂D, then the set $\Phi^{-1}(\mathbf{p})$, the preimage of $\mathbf{p}$ in D, is discrete and, finally, at each $\mathbf{x} \in \mathbf{\Phi}^{-1}(\mathbf{p})$, the Jacobian

$$J_{\mathbf{\Phi}}(\mathbf{x}) = \det\left(\frac{\partial \Phi_i}{\partial x_j}\right)$$

does not vanish. Then the degree of $\mathbf{\Phi}$ with respect to $\mathbf{p}$ and D is

$$\deg(\mathbf{p}, \mathbf{\Phi}, D) = \sum_{\substack{\mathbf{\Phi}(\mathbf{x})=\mathbf{p} \\ \mathbf{x}\in D}} \operatorname{sign} J_{\mathbf{\Phi}}(\mathbf{x})$$

where $\operatorname{sign} J_{\mathbf{\Phi}}(\mathbf{x})$ is the signum of $J_{\mathbf{\Phi}}(\mathbf{x})$.

If $\deg(\mathbf{p}, \mathbf{\Phi}, D) \neq 0$, then there are solutions of the equation $\mathbf{\Phi}(\mathbf{x}) = \mathbf{p}$ in D. If $\mathbf{p} \notin \mathbf{\Phi}(D)$ then $\deg(\mathbf{p}, \mathbf{\Phi}, D) = 0$ and so $\deg(\mathbf{p}, \mathbf{\Phi}, D)$ determines, in a certain way, the number of solutions of the equation $\mathbf{\Phi}(\mathbf{x}) = \mathbf{p}$.

If there are points $\mathbf{x}$ at which $\mathbf{\Phi}(\mathbf{x}) = \mathbf{p}$ and $J_{\mathbf{\Phi}}(\mathbf{x}) = 0$, then we can introduce the degree of the map using the limit passage. We can always take a sequence of points $\mathbf{p}_k \to \mathbf{p}$ such that $J_{\mathbf{\Phi}}(\mathbf{x}) \neq 0$ at any $\mathbf{x} \in \mathbf{\Phi}^{-1}(\mathbf{p}_k)$;

the degree of $\mathbf{\Phi}$ is now defined by

$$\deg(\mathbf{p}, \mathbf{\Phi}, D) = \lim_{k\to\infty} \deg(\mathbf{p}_k, \mathbf{\Phi}, D).$$

It is shown that this number does not depend on the choice of the sequence $\{\mathbf{p}_k\}$ and also characterizes the number of solutions of the equation $\mathbf{\Phi}(\mathbf{x}) = \mathbf{p}$ in D.

The next step of the theory is to state it for $\mathbf{\Phi}(\mathbf{x})$ being of $C(\overline{D})$ (each of the components of $\mathbf{\Phi}(\mathbf{x})$ being of $C(\overline{D})$). This is done by using a limit passage. Namely, for $\mathbf{\Phi}(\mathbf{x})$, there is a sequence $\{\mathbf{\Phi}_k(\mathbf{x})\}$ such that $\mathbf{\Phi}_k(\mathbf{x}) \in C^{(1)}(\overline{D})$ and each component of $\mathbf{\Phi}_k(\mathbf{x})$ converges uniformly on $\overline{D}$ to a corresponding component of $\mathbf{\Phi}(\mathbf{x})$. Then as is shown, there exists

$$\lim_{k\to\infty} \deg(\mathbf{p}, \mathbf{\Phi}_k, D)$$

which does not depend on the choice of $\{\mathbf{\Phi}_k(\mathbf{x})\}$; it is, by definition, the degree of $\mathbf{\Phi}(x)$ with respect to $\mathbf{p}$ and D.

As there is a one-to-one correspondence between $\mathbb{R}^n$ and n-dimensional real Banach space, the notion of degree of a map is transferred to continuous maps in the latter space. Moreover, it is seen how it can be determined for a continuous map whose range is a finite-dimensional subspace of a Banach space.

In the case of general operator in a Banach space, the notion was extended to operators of the form $I + F$ with a compact operator F on a real Banach space X by J. Leray and J. Schauder [15]. To do this, they introduce an approximate operator as follows.

Let D be a bounded open domain in X with the boundary ∂D. As F is a compact operator, $F(\overline{D})$, the image of $\overline{D}$, is compact. So, by the Hausdorff criterion on compactness, there is a finite ε-net $N_\varepsilon = \{x_k \mid x_k \in F(\overline{D}); k = 1, \dots, n\}$, such that for every $x \in \overline{D}$ there is an integer k such that $\|F(x) - x_k\| < \varepsilon$. Finally, the approximate operator F_ε is defined by

$$F_\varepsilon(x) = \frac{\sum_{k=1}^n \mu_k(x) x_k}{\sum_{k=1}^n \mu_k(x)}, \qquad x \in \overline{D}$$

where $\mu_k(x) = 0$ if $\|F(x) - x_k\| > \varepsilon$ and $\mu_k = \varepsilon - \|F(x) - x_k\|$ if $\|F(x) - x_k\| \le \varepsilon$. This operator is called the Schauder projection operator.

It is easily seen that the range of $F_\varepsilon(x)$ is a domain in a finite dimensional subspace X_n of X, the operator F_ε is continuous, and, moreover,

$$\|F(x) - F_\varepsilon(x)\| \le \varepsilon$$

when $x \in \overline{D}$.

By the above, we can introduce the degree of $I + F_\varepsilon$ with respect to p and $D_n = D \cap X_n$ if $p \notin (I + F_\varepsilon)(\partial D_n)$. As is shown in Schwartz [21], for sufficiently small $\varepsilon > 0$ the degree $\deg(p, I + F_\varepsilon, D_n)$ is the same and thus it is defined as the degree of the operator $I + F$ with respect to p and D.

The following properties of the degree of an operator $I+F$ with compact operator F hold:

1. If $x + F(x) \neq p$ in $\overline{D}$, then $\deg(p, I+F, D) = 0$;
2. if $\deg(p, I+F, D) \neq 0$, then in D there is at least one solution to the equation $x + F(x) = p$;
3. $\deg(p, I, D) = +1$ if $p \in D$;
4. if $D = \cup_i D_i$ where the family $\{D_i\}$ is disjoint and $\partial D_i \subset \partial D$, then
$$\deg(p, I+F, D) = \sum_i \deg(p, I+F, D_i);$$
5. $\deg(p, I+F, D)$ is continuous with respect to p and F;
6. (invariance under homotopy) Let $\Phi(x,t) = x + \Psi(x,t)$. Assume that for every $t \in [a,b]$ the operator $\Phi(x,t)$ is compact with respect to $x \in X$ and continuous in $t \in [a,b]$ uniformly with respect to $x \in \overline{D}$. Then the operators $\Psi_a = \Psi(\cdot, a)$ and $\Psi_b = \Psi(\cdot, b)$ are said to be compact homotopic. Let Ψ_a and Ψ_b be compact homotopic and $p \neq x + \Psi(x,t)$ for every $x \in \partial D$ and $t \in [a,b]$; then
$$\deg(p, I+\Psi_a, D) = \deg(p, I+\Psi_b, D).$$

The sixth and third properties give a result that is frequently used to establish existence of solution of the equation
$$x + F(x) = 0. \tag{3.7.1}$$
We formulate it as

Lemma 3.7.1. Assume $F(x)$ is a compact operator in a Banach space X and the equation $x + tF(x) = 0$ has no solutions on a sphere $\|x\| = R$ for any $t \in [a,b]$. Then in the ball $B = \{x \mid \|x\| < R\}$ there exists at least one solution to (3.7.1) and
$$\deg(0, I+F, B) = +1.$$

In the next section we demonstrate an application of the lemma.

3.8 Steady-State Flow of Viscous Liquid

Following I.I. Vorovich and V.I. Yudovich [27], we consider the steady-state flow of a viscous incompressible liquid described by the Navier–Stokes equations
$$\nu \Delta \mathbf{v} = (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla p + \mathbf{f}, \tag{3.8.1}$$

$$\nabla \cdot \mathbf{v} = 0. \tag{3.8.2}$$

Let $\nu > 0$. We are treating a problem with boundary condition

$$\mathbf{v}\big|_{\partial\Omega} = \boldsymbol{\alpha}. \tag{3.8.3}$$

From now on, we assume:

(i) Ω is a bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ whose boundary $\partial\Omega$ consists of r closed curves or surfaces S_k, $k = 1, \dots, r$ with continuous curvature.

(ii) There is a continuously differentiable vector-function

$$\mathbf{a}(\mathbf{x}) = (a_1(\mathbf{x}), a_2(\mathbf{x}), a_3(\mathbf{x}))$$

such that

$$a_k(\mathbf{x}) \in C^{(1)}(\overline{\Omega}), \qquad \nabla \cdot \mathbf{a} = 0 \text{ in } \Omega, \qquad \mathbf{a}\big|_{\partial\Omega} = \boldsymbol{\alpha}.$$

(iii) On each S_k, $k = 1, \dots, r$, we have

$$\int_{S_k} \boldsymbol{\alpha} \cdot \mathbf{n}\, dS = 0 \tag{3.8.4}$$

where $\mathbf{n}$ is the unit outward normal at a point of S_k.

We note that the condition

$$\sum_{k=1}^{r} \int_{S_k} \boldsymbol{\alpha} \cdot \mathbf{n}\, dS = 0$$

is necessary for solvability of the problem.

Let $H(\Omega)$ be the completion of the set $S^0(\Omega)$ of all smooth solenoidal vector-functions $\mathbf{u}(\mathbf{x})$ satisfying the boundary condition, in the norm induced by the scalar product

$$(\mathbf{u}, \mathbf{v})_{H(\Omega)} = \int_{\Omega} \nabla\mathbf{u} \cdot \nabla\mathbf{v}\, d\Omega \equiv \int_{\Omega} \operatorname{rot}\mathbf{u} \cdot \operatorname{rot}\mathbf{v}\, d\Omega$$

and so each of the components of $\mathbf{u}(\mathbf{x}) \in H(\Omega)$ is of $W^{1,2}(\Omega)$. Thus in the three dimensional case, the imbedding operator of $H(\Omega)$ into $(L^p(\Omega))^3$ is continuous when $1 \le p \le 6$ and compact when $1 \le p < 6$; in the two dimensional case, the imbedding operator is compact into $(L^p(\Omega))^2$ for any $1 \le p < \infty$.

We assume

(iv) $f_k(\mathbf{x}) \in L^p(\Omega)$, $p \ge 6/5$ in the three dimensional case ($k = 1, 2, 3$), $p > 1$ in the two dimensional case ($k = 1, 2$).

Definition 3.8.1. $\mathbf{v}(\mathbf{x}) = \mathbf{a}(\mathbf{x}) + \mathbf{u}(\mathbf{x})$ is called a generalized solution to the problem (3.8.1)–(3.8.3) if $\mathbf{u}(\mathbf{x}) \in H(\Omega)$ and satisfies the integro-differential equation

$$\nu(\mathbf{u}, \mathbf{\Phi})_{H(\Omega)} = -\int_\Omega [(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{\Phi} + (\mathbf{u}\cdot\nabla)\mathbf{a}\cdot\mathbf{\Phi} + (\mathbf{a}\cdot\nabla)\mathbf{u}\cdot\mathbf{\Phi} + \\ + (\mathbf{a}\cdot\nabla)\mathbf{a}\cdot\mathbf{\Phi} + \nu \operatorname{rot}\mathbf{a}\cdot\operatorname{rot}\mathbf{\Phi} + \mathbf{f}\cdot\mathbf{\Phi}]\, d\Omega \tag{3.8.5}$$

for any $\mathbf{\Phi} \in H(\Omega)$.

It is easily seen that if $\mathbf{a}(\mathbf{x})$ and $\mathbf{u}(\mathbf{x})$ belong to $C^{(2)}(\overline{\Omega})$ then $\mathbf{v}(\mathbf{x})$ is a classical solution to the problem (3.8.1)–(3.8.3).

Note that there are infinitely many vectors $\mathbf{a}(\mathbf{x})$ satisfying the assumption (ii) if there is one, but the set of generalized solutions does not depend on the choice of $\mathbf{a}(\mathbf{x})$.

To use Lemma 3.7.1, we reduce equation (3.8.5) to the operator form $\mathbf{u} + F(\mathbf{u}) = 0$, defining F with use of the Riesz representation theorem from the equality

$$\nu(F(\mathbf{u}), \mathbf{\Phi})_{H(\Omega)} = \int_\Omega [(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{\Phi} + (\mathbf{u}\cdot\nabla)\mathbf{a}\cdot\mathbf{\Phi} + (\mathbf{a}\cdot\nabla)\mathbf{u}\cdot\mathbf{\Phi} + \\ + (\mathbf{a}\cdot\nabla)\mathbf{a}\cdot\mathbf{\Phi} + \nu \operatorname{rot}\mathbf{a}\cdot\operatorname{rot}\mathbf{\Phi} + \mathbf{f}\cdot\mathbf{\Phi}]\, d\Omega. \tag{3.8.6}$$

The estimates needed to prove that the right-hand side of (3.8.6) is a continuous linear functional in $H(\Omega)$ with respect to $\mathbf{\Phi}$ follow from traditional estimates of the terms using the Hölder inequality. But we now show a sharper result; namely,

Lemma 3.8.1. *F is a completely continuous operator in $H(\Omega)$.*

Proof. Let $\{\mathbf{u}_n(x)\}$ be a weakly convergent sequence in $H(\Omega)$. Then it converges strongly in $(L^4(\Omega))^k$ ($k = 2$ or 3). From (3.8.6), we get

$$\begin{aligned} &\nu|(F(\mathbf{u}_m) - F(\mathbf{u}_n), \mathbf{\Phi})_{H(\Omega)}| = \\ &\quad = \Big| \int_\Omega \{[(\mathbf{u}_m - \mathbf{u}_n)\cdot\nabla]\mathbf{u}_m\cdot\mathbf{\Phi} - (\mathbf{u}_n\cdot\nabla)(\mathbf{u}_m - \mathbf{u}_n)\cdot\mathbf{\Phi} + \\ &\qquad + [(\mathbf{u}_m - \mathbf{u}_n)\cdot\nabla]\mathbf{a}\cdot\mathbf{\Phi} + (\mathbf{a}\cdot\nabla)(\mathbf{u}_m - \mathbf{u}_n)\cdot\mathbf{\Phi}\}\, d\Omega \Big| \\ &\quad \le M\|\mathbf{u}_m - \mathbf{u}_n\|_{L^4(\Omega)}\|\mathbf{\Phi}\|_{H(\Omega)} \end{aligned}$$

with a constant M which does not depend on m, n, or $\mathbf{\Phi}$. Setting

$$\mathbf{\Phi} = F(\mathbf{u}_m) - F(\mathbf{u}_n)$$

in the inequality, we obtain

$$\nu\|F(\mathbf{u}_m) - F(\mathbf{u}_n)\|_{H(\Omega)} \le M\|\mathbf{u}_m - \mathbf{u}_n\|_{(L^4(\Omega))^k} \to 0$$

when $m, n \to \infty$, and so F is completely continuous. $\square$

From Definition 3.8.1 it follows that

Lemma 3.8.2. A generalized solution of the problem under consideration in the sense of Definition 3.8.1 satisfies the operator equation

$$\mathbf{u} + F(\mathbf{u}) = 0; \tag{3.8.7}$$

conversely, a solution to (3.8.7) is a generalized solution of the problem.

By Lemma 3.7.1, it now suffices to show that all solutions of the equation $\mathbf{u}+tF(\mathbf{u}) = 0$, for all $t \in [0, 1]$, lie in a sphere $\|\mathbf{u}\|_{H(\Omega)} \leq R$ for some $R < \infty$. First we show this in the simpler case of homogeneous boundary condition (3.8.3). Here $\boldsymbol{\alpha} = 0$ and thus $\mathbf{a}(\mathbf{x}) = 0$. □

Theorem 3.8.1. The problem (3.8.1)–(3.8.3) with $\boldsymbol{\alpha} = 0$ has at least one generalized solution in the sense of Definition 3.8.1. Each generalized solution $\mathbf{u}(\mathbf{x})$ is bounded, $\|\mathbf{u}\|_{H(\Omega)} < R$ for some $R < \infty$ and the degree of $I + F$ with respect to 0 and $D = \{\mathbf{u} \in H(\Omega) \mid \|\mathbf{u}\| < R\}$ is $+1$.

Proof. As was said, it suffices to show an a priori estimate for solutions to the equation $\mathbf{u} + tF(\mathbf{u}) = 0$ for $t \in [0, 1]$. For a solution, there holds the identity

$$(\mathbf{u} + tF(\mathbf{u}), \mathbf{u})_{H(\Omega)} = 0$$

or, the same,

$$\nu(\mathbf{u}, \mathbf{u})_{H(\Omega)} + t \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{u}\, d\Omega = -t \int_\Omega \mathbf{f} \cdot \mathbf{u}\, d\Omega.$$

Integration by parts gives

$$\begin{aligned} \int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{u}\, d\Omega &= \frac{1}{2} \int_\Omega \sum_k u_k \frac{\partial}{\partial x_k} (\mathbf{u} \cdot \mathbf{u})\, d\Omega \\ &= -\frac{1}{2} \int_\Omega (\mathbf{u} \cdot \mathbf{u})(\nabla \cdot \mathbf{u})\, d\Omega = 0 \end{aligned} \tag{3.8.8}$$

since $\nabla \cdot \mathbf{u} = 0$ and thus, for a solution $\mathbf{u}$, we get

$$|\nu(\mathbf{u}, \mathbf{u})_{H(\Omega)}| = \left| t \int_\Omega \mathbf{f} \cdot \mathbf{u}\, d\Omega \right| \leq \frac{\nu R}{2} \|\mathbf{f}\|_{L^p(\Omega)} \|\mathbf{u}\|_{H(\Omega)}$$

with some constant R, or

$$\|\mathbf{u}\|_{H(\Omega)} < R.$$

This completes the proof. □

Now we consider the more complicated case of nonhomogeneous boundary conditions (3.8.3). We need some auxiliary results.

Let ω_ε be a domain in $\overline{\Omega}$ which consists of points covered by all inward normals to $\partial\Omega$ of the length ε. For sufficiently small $\varepsilon > 0$, these normals do not intersect and thus in ω_ε we can use a coordinate system pointing out for a $\mathbf{x} \in \omega_\varepsilon$ a point Q on $\partial\Omega$ and a number s, the distance from Q to $\mathbf{x}$ along the corresponding normal. So for a function $g(\mathbf{x})$ given on ω_ε, we write down $g(s, Q)$.

Lemma 3.8.3. There is a solenoidal vector function $\mathbf{a}_\varepsilon(\mathbf{x}) \in (C^{(1)}(\overline{\Omega}))^k$ such that $\mathbf{a}_\varepsilon(\mathbf{x}) = 0$ in $\Omega \setminus \omega_\varepsilon$,

$$\mathbf{a}_\varepsilon(\mathbf{x})\big|_{\partial\Omega} = \boldsymbol{\alpha}, \qquad \text{and} \qquad |\mathbf{a}_\varepsilon(\mathbf{x})| \le M_1/\varepsilon \text{ in } \overline{\Omega} \tag{3.8.9}$$

with a constant M_1 not depending on ε.

Proof. Let us introduce a function $q(\mathbf{x})$ by

$$q(s, Q) = \begin{cases} (\varepsilon^2 - s^2)^2/\varepsilon^4, & 0 \le s \le \varepsilon, \\ 0, & s > \varepsilon. \end{cases}$$

Let $\mathbf{a}(\mathbf{x})$ be a solenoidal vector-function satisfying the assumption (ii) of the beginning of the section. Under the taken assumptions, there is a vector-function $\mathbf{p}(\mathbf{x})$ such that

$$\mathbf{a}(\mathbf{x}) = \operatorname{rot} \mathbf{p}(\mathbf{x}).$$

It is seen that the vector function $\mathbf{a}_\varepsilon(\mathbf{x}) = \operatorname{rot}(q\mathbf{p})$ is needed.

Note that in the plane case, this is a vector $(0, 0, q\psi)$ where $\psi(x_1, x_2)$ is the flow function of $\mathbf{a}(\mathbf{x})$. □

Lemma 3.8.4. For $\mathbf{u} \in H(\Omega)$, we have

$$\int_{\omega_\varepsilon} |\mathbf{u}|^2 \, d\Omega \le M_2^2 \varepsilon^2 \int_{\omega_\varepsilon} \sum_{i,j} \left| \frac{\partial u_i}{\partial x_j} \right|^2 d\Omega \tag{3.8.10}$$

with a constant M_2 not depending on $\mathbf{u}$ or ε.

Proof. We show (3.8.10) for a smooth function. The limit passage will prove the general case. So for points of ω_ε we have

$$\mathbf{u}(s, Q) = \int_0^s \frac{\partial \mathbf{u}(t, Q)}{\partial t} \, dt.$$

By the Cauchy inequality

$$\begin{aligned} \int_0^\varepsilon |\mathbf{u}(t, Q)|^2 \, dt &= \int_0^\varepsilon \left| \int_0^s \frac{\partial \mathbf{u}(t, Q)}{\partial t} \, dt \right|^2 ds \\ &\le \int_0^\varepsilon s \int_0^s \left| \frac{\partial \mathbf{u}(t, Q)}{\partial t} \right|^2 dt \, ds \\ &\le \frac{\varepsilon^2}{2} \int_0^\varepsilon \left| \frac{\partial \mathbf{u}(t, Q)}{\partial t} \right|^2 dt. \end{aligned}$$

It is easily seen that for any $g(\mathbf{x})$

$$m_1 \int_0^\varepsilon \int_{\partial\Omega} g^2(s,Q)\,ds\,dS \le \int_{\omega_\varepsilon} g^2\,d\Omega \le m_2 \int_0^\varepsilon \int_{\partial\Omega} g^2(s,Q)\,ds\,dS$$

and so

$$\begin{aligned}\int_{\omega_\varepsilon} |\mathbf{u}|^2\,d\Omega &\le m_2 \int_{\partial\Omega}\int_0^\varepsilon |\mathbf{u}(s,Q)|^2\,ds\,dS \\ &\le m_2 \int_{\partial\Omega} \frac{\varepsilon^2}{2} \int_0^\varepsilon \left|\frac{\partial \mathbf{u}}{\partial t}\right|^2 dt\,dS \\ &\le \frac{m_2}{2m_1}\varepsilon^2 \int_{\omega_\varepsilon} \sum_{i,j} \left|\frac{\partial u_i}{\partial x_j}\right|^2 d\Omega.\end{aligned}$$

□

To apply degree theory to the problem under consideration, it remains to establish

Lemma 3.8.5. All solutions of the equation

$$\mathbf{u} + tF(\mathbf{u}) = 0 \tag{3.8.11}$$

for all $t \in [0,1]$, are in a ball $\|\mathbf{u}\|_{H(\Omega)} < R$ whose radius R depends only on $\mathbf{f}$, $\partial\Omega$, $\mathbf{a}$, and ν.

Proof. Suppose that the set of solutions to (3.8.11) is unbounded. This means there is a sequence $\{t_k\} \subset [0,1]$ and a corresponding sequence $\{\mathbf{u}_k\}$ such that $\mathbf{u}_k + t_k F(\mathbf{u}_k) = 0$ and

$$\|\mathbf{u}_k\|_{H(\Omega)} \to \infty \text{ as } k \to \infty. \tag{3.8.12}$$

Without loss of generality, we can consider $\{t_k\}$ to be convergent to $t_0 \in [0,1]$ and, moreover, the sequence $\{\mathbf{u}_k^*\}$, $\mathbf{u}_k^* = \mathbf{u}_k/\|\mathbf{u}_k\|_{H(\Omega)}$, to be weakly convergent to an element $\mathbf{u}_0 \in H(\Omega)$ since $\{\mathbf{u}_k^*\}$ is bounded.

Let us consider the identity $(\mathbf{u}_k + t_k F(\mathbf{u}_k), \mathbf{u}_k) = 0$, namely,

$$\begin{aligned}-\nu\|\mathbf{u}_k\|^2_{H(\Omega)} = t_k &\int_{\omega_\varepsilon} (\mathbf{a}_\varepsilon \cdot \nabla)\mathbf{u}_k \cdot \mathbf{u}_k\,d\Omega + \\ &+ t_k \int_\Omega [(\mathbf{a}_\varepsilon \cdot \nabla)\mathbf{a}_\varepsilon \cdot \mathbf{u}_k + \nu \operatorname{rot} \mathbf{a}_\varepsilon \cdot \operatorname{rot} \mathbf{u}_k + \mathbf{f} \cdot \mathbf{u}_k]\,d\Omega\end{aligned} \tag{3.8.13}$$

which is valid because of (3.8.8) and a similar equality

$$\int_{\omega_\varepsilon} (\mathbf{u}_k \cdot \nabla)\mathbf{a}_\varepsilon \cdot \mathbf{u}_k\,d\Omega = 0.$$

The first integral on the right-hand side of (3.8.13) is a weakly continuous functional with respect to $\mathbf{u}_k$, and for the second integral we have

$$\left| \int_\Omega [(\mathbf{a}_\varepsilon \cdot \nabla)\mathbf{a}_\varepsilon \cdot \mathbf{u}_k + \nu \operatorname{rot} \mathbf{a}_\varepsilon \cdot \operatorname{rot} \mathbf{u}_k + \mathbf{f} \cdot \mathbf{u}_k] \, d\Omega \right| \le M_3 \|\mathbf{u}_k\|_{H(\Omega)}$$

where M_3 does not depend on $\mathbf{u}_k$. Dividing both sides of (3.8.13) by $\|\mathbf{u}_k\|^2_{H(\Omega)}$, it follows that

$$-\nu = t_0 \int_{\omega_\varepsilon} (\mathbf{a}_\varepsilon \cdot \nabla)\mathbf{u}_0 \cdot \mathbf{u}_0 \, d\Omega. \tag{3.8.14}$$

We note that this holds for any small positive $\varepsilon < \varepsilon_0$ with a fixed ε_0 for which the above construction of the frame for ω_{ε_0} is valid. To prove it, take $\varepsilon = \eta < \varepsilon_0$

$$\mathbf{w}_k = \mathbf{u}_k + \mathbf{a}_{\varepsilon_0} - \mathbf{a}_\eta$$

and consider the identity

$$(\mathbf{u}_k + t_k F(\mathbf{u}_k), \mathbf{w}_k)_{H(\Omega)} = 0$$

which takes the form

$$\begin{aligned} -\nu \|\mathbf{w}_k\|^2_{H(\Omega)} = t_k \int_{\omega_\eta} (\mathbf{a}_\eta \cdot \nabla)\mathbf{w}_k \cdot \mathbf{w}_k \, d\Omega + \\ + t_k \int_\Omega [(\mathbf{a}_\eta \cdot \nabla)\mathbf{a}_\eta \cdot \mathbf{w}_k + \nu \operatorname{rot} \mathbf{a}_\eta \cdot \operatorname{rot} \mathbf{w}_k + \mathbf{f} \cdot \mathbf{w}_k] \, d\Omega. \end{aligned}$$

Divide this equality by $\|\mathbf{u}_k\|^2_{H(\Omega)}$ term by term. Consider the sequence

$$\mathbf{w}_k^* = \mathbf{u}_k^* + (\mathbf{a}_\varepsilon - \mathbf{a}_\eta)/\|\mathbf{u}_k\|_{H(\Omega)}.$$

Since $\|\mathbf{u}_k\|_{H(\Omega)} \to \infty$, we have $(\mathbf{a}_\varepsilon - \mathbf{a}_\eta)/\|\mathbf{u}_k\|_{H(\Omega)} \to 0$ strongly. Since $\|\mathbf{u}_k^*\|_{H(\Omega)} = 1$, we have that $\|\mathbf{w}_k^*\|_{H(\Omega)} \to 1$. Besides, it is clear that $\mathbf{w}_k^* \to \mathbf{u}_0$ weakly and thus we get the needed equality (3.8.14) again.

Now we show that the limit of the integral on the right-hand side of (3.8.14) is zero. Thanks to (3.8.9) and (3.8.10), we obtain

$$\left| \int_{\omega_\varepsilon} (\mathbf{a}_\varepsilon \cdot \nabla)\mathbf{u}_0 \cdot \mathbf{u}_0 \, d\Omega \right| \le M_1 M_2 \int_{\omega_\varepsilon} |\operatorname{rot} \mathbf{u}_0|^2 \, d\Omega \to 0$$

as $\varepsilon \to 0$. Since $\nu > 0$, we have a contradiction which completes the proof. □

Now we can formulate

Theorem 3.8.2. Under assumptions (i)–(iv), there exists at least one generalized solution of the problem (3.8.1)–(3.8.3) in the sense of Definition 3.8.1. All generalized solutions of the problem are bounded in the energy space and the degree of the operator $I + F$ of the problem with respect to zero and a ball about zero with sufficiently large radius is $+1$.

Problem 3.8.1. Check which of the assumptions (i)–(iv) are not necessary in proving Theorem 3.8.1.