ERRATA & NOTES
Functional Analysis in Mechanics

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- Page 1. $k = 1, 2, \ldots$ should read $k = 0, 1, \ldots$.
- Page 11, 2nd displayed equation. Upper limit of integral should read π instead of 2π.
- Page 22, 5th line from top. Missed closing parenthesis after y_n.
- Page 22, 8th line from top. "X, Y" should read "of X, Y".
- Page 33, first line of (1.10.4). $y''(t)$ should read $y_n''(t)$.
- Page 36, first line of the equation array near the top. Missing superscript "2" after the large closing parenthesis.
- Page 39, equation (1.10.14). Insert negative signs on both sides:

$$-(w_1, w_2) = -\iint\limits_{\Omega} D^{\alpha\beta\gamma\delta}\rho_{\gamma\delta}(w_1)\rho_{\alpha\beta}(w_2)\,dx\,dy \qquad (1.10.14)$$

- Page 45, equation (1.10.26). Should read

$$-(\mathbf{u}, \mathbf{v}) = -\int_{\Omega} c^{ijkl}\epsilon_{kl}(\mathbf{u})\epsilon_{ij}(\mathbf{v})\,d\Omega. \qquad (1.10.26)$$

- Page 50, 3rd line above last displayed equation. "minimal index" should read "maximal index".
- Page 52, bottom. Remark. The equation $x = Ax$ could have been stated as $x = A(x)$, since the contraction mapping theorem applies to nonlinear operators as well as mappings on general metric spaces.
- Page 57, middle, "The values of q ...". Remark. Here it is understood (since we can speak of an operator norm only for a *linear* operator) that the reader is to consider only the case in which all $c_i = 0$.

- Page 60. Remark. In textbooks it is more frequent that the first integral term in (1.14.11) is called the work of internal forces when taken with a minus sign. Then the principle of virtual displacements reads that the sum of the work of internal and external forces on any virtual displacements is zero. So the paragraph immediately following Definition 1.14.1 could be reworded as follows:

  > We can also obtain (1.14.11) from the principle of virtual displacements (work). This asserts that in the state of equilibrium, the sum of the work of internal forces (which is now the variation of internal energy with sign "-") and external forces on all virtual (admissible) displacements is zero.

- Page 62, equation (1.14.14). Should read

$$\int_\Omega F(x,y)w_i(x,y)\,dx\,dy + \sum_{k=1}^{m} F_k w_i(x_k,y_k) + \int_\gamma f(s)w_i(x,y)\,ds = 0 \tag{1.14.14}$$

- Page 68, proof of Hausdorff criterion. Remark. A subsequence is a portion of the initial sequence preserving ordering of the members. In the sufficiency proof we must choose a subsequence in such a way that $i_1 < i_2 < i_3 < \cdots$. This is possible since on each step during the choice we have an infinite number of appropriate members.

- Page 83, near the top. Remark. Here we make use of Theorem 2.3.2.

- Page 74, displayed formula between (1.17.5) and (1.17.6). Change $\mathbf{y}_{\Delta_k}$ to $\mathbf{y}_{\Delta_{k_1}}$.

- Page 101, end of the first paragraph of the proof. "by definition of weak convergence" should read "by the definition of a weakly Cauchy sequence". Alternatively, one could replace the phrase in question by

  > since a weakly Cauchy sequence is bounded, the numerical sequence $\{(x_n, y_0)\}$ is bounded as well.

  Remark. Although the proof is given in a *real* Hilbert space, the theorem is valid in a complex space as well. For the complex case one need only supply absolute value signs around inner products.

- Page 107, Remark 1.23.1. "(1.23.7)" should read "(1.23.6)".

- Page 115, three lines below equation (1.26.2). "Then (1.26.1)" should read "Then (1.26.2)".

- Page 117, displayed equation at end of proof. Missing asterisk on "P_{k-1}".

- Page 123, fourth line from bottom. "$g_n, \ldots, g_n$" should read "$g_1, \ldots, g_n$".

- Page 140, 3rd line after the end of the proof. "ball is compact" should read "ball is precompact".

- Page 141, third line after the end of the proof. Delete the phrase "in $C(0, 1)$ and".

- Page 153, proof of Theorem 2.10.3. In the first line of the first displayed formula, $R(\lambda A)$ should read $R(\lambda, A)$.

- Page 156. The last formula on this page should be finished with a period. The first line on page 157 should start as: "Since ... as well, we have proved the needed property for the term in parenthesis."

- Page 155. In Section 2.11 (page 155) we change terminology for eigenvalues when we call μ from equation (2.11.1) the eigenvalue when it corresponds to nontrivial solutions of equation

  $$(I - \mu A)x = 0.$$

  In the generalized setup of the eigenfrequency problem for a membrane which is governed by the equation

  $$\Delta u = -\omega^2 u,$$

  the term Δu corresponds to operator I, whereas the right-hand side term u corresponds to operator A of (2.11.1). Thus $\mu =$ is equal to the squared eigenfrequency of the membrane, ω^2.

- Page 157. In the formulation of Theorem 2.11.1 it is more common to use the notation $\{0\}$ for the space consisting of the single element zero, so with this notation, the formula for the intersection of the subspaces is $H(\mu_k) \bigcap H(\mu_n) = \{0\}$.

- Page 159. Last line of Corollary 2.11.2 should read "where $x \in M$ is a solution to (2.11.6)."

- Page 167, proof of Theorem 2.14.1. "Since $\|A\| = \sup_{\|x\|\le 1} |(Ax, x)|$" should read "Since $\|A\| = \sup_{\|x\|\le 1} |(Ax, x)| \ne 0$".

- Page 175. First equation display should have "$n + 1$" instead of "n" as subscripts on λ and μ (3 occurrences).

- Page 180, first line. "A^{-1}" should read "A".

- Page 182, first displayed equation. "$F_x(x, \mu_0)$" should read "$F_x(x_0, \mu_0)$".

- Page 185. Theorem 3.3.1, 2nd line of proof. Remark. "minimum", more generally, could read "extremum".

- Page 186. Lines 9–10. "is thus clearly seen" should read "is seen similarly".

- Page 187. Theorem 3.3.3, part (ii). The statement should read "(ii) any minimizing sequence $\{x_n\}$ of $\Psi(x)$ contains a ...".
- Page 222, near the bottom. "$\mathbf{n}$ to Ω" should read "$\mathbf{n}$ to $\partial\Omega$".

An extra section

A Spanish edition of *Functional Analysis in Mechanics* is currently under preparation by Prof. Francisco Caicedo (Dept. of Mathematics, National University of Colombia). This edition will conclude with an epilogue, the purpose of which is to place the material into proper perspective for the engineering reader. The English version of the additional text appears below.

Epilogue to the Spanish edition

Functional analysis is relatively difficult for one whose background is limited to standard courses in engineering mathematics. The busy practitioner may wonder whether it makes sense to tackle this seemingly abstract area. What sorts of useful information can be gleaned from all these axioms, definitions, and theorems? Indeed, the applicability of any unfamiliar branch of mathematics can be hard to judge at first glance. Now that the reader has spent at least some time with the material, however, we are in a much better position to advance a case in favor of functional analysis. Let us begin by discussing the mathematical objects and tools that engineers use in mechanical design.

Many standard mechanics problems can be solved numerically. In view of this, and given the ever-increasing sophistication of computer hardware and software, we might expect that less and less mathematical/scientific capability will continue to be required from the average engineer — culminating in a situation where computers handle almost every step of the design process. The happy human operator could then forget about such burdensome things as physical laws and the assumptions that underlie numerical solution methods! However, a more sober outlook must account for the fact that many engineering problems are in some sense nonstandard: they fall well outside the range of problems for which any given software package was developed and tested. For these problems, unfortunately, simple and "direct" calculation approaches can fail miserably. So the competent designer must be ready to carefully interpret numerical output and render a sound judgement as to its reliability. This, in turn, requires familiarity with the general properties of the problem under investigation.

Every model of mechanics should be regarded as approximate. Even the most precise models still neglect the atomic structure of materials, the details of the reactions between bodies, the effects of complicated surface imperfections, etc. It is therefore fallacious to assert that a certain mathematical boundary value problem must be well-posed because the physical problem that it "describes" seems well-posed in practice. For example, a body under load and free of geometric restrictions can be in equilibrium. But any negligibly small change in the load, making it non-self-balanced, brings us to an equilibrium problem that has no solution. Having formulated a boundary value problem then, we should study its properties not only by comparing solutions with experimental data (which, by the way, is possible under very restrictive conditions and in large part only on boundary surfaces) but also through the use of rigorous mathematical tools.

In addition, all modern numerical methods used to investigate the motions and strain states of real bodies are based on a two-step procedure. First, a continuous model, represented by a boundary value problem for a system of differential or integro-differential equations, is composed. Inherent in this are limiting and averaging processes based on some set of imperfect constitutive laws for the material. Second, the boundary value problem is discretized. In some sense this is a step backward: differential equations are converted to a system of simpler — normally algebraic — equations which can be solved (again, approximately) using a finite number of elementary arithmetic operations. The discrepancy between the solutions of the two corresponding problems may be significant, even when both can be well-posed.

In continuum mechanics we can regard the correspondences between loads or other external factors and strain states or velocity fields as "functions" in some sense. These are more general than the functions of elementary mathematics, however; they can provide mappings between sets of vector or tensor functions residing in infinite dimensional spaces. At the same time, problems solvable by computers are necessarily formulated in finite dimensional spaces, not to mention the fact that machine calculations themselves are performed using finite precision. Common sense dictates that we cannot accurately approximate all the elements of an infinite dimensional space using only a finite number of elements. General theory also confirms that there exist situations in which accurate approximation is impossible via rote use of standard programs. Enlightened use of such programs therefore requires enough theoretical background to adequately distinguish these situations from those in which computational methods can be expected to bring accurate results.

Questions involving the composition of mechanical models and discretization of corresponding boundary value problems now fall within the applied disciplines such as physics and mechanics. Experts in these areas are often more successful than mathematicians in performing these tasks, because they can invoke the kind of intuition that comes from direct experience with numerous models and the real objects they are intended to represent. But the study of mathematical models and their properties, as well as the behavior of associated numerical solution schemes, has become the province of the mathematician (unlike the situation in the days of Newton, Euler, or Lagrange, when certain individuals could stand at the forefront of both mathematics and the physical sciences).

The theoretical study of mechanics problems — their well-posedness and other properties, as well as the legitimacy of numerical solution methods — constitutes a substantial portion of functional analysis. This is not the classical mathematical analysis normally taught to engineers. The difference lies in the fact that the boundary value problems of continuum mechanics are usually described in infinite dimensional spaces where classical analysis has limited reach. Nonetheless, functional analysis is not independent of the calculus introduced by Newton and Leibnitz. Rather it serves to extend calculus in combination with the calculus of variations, the methods of classical algebra, and other useful branches of classical mathematics. Functional analysis has elaborated its own terminology and tools, many of which were borrowed from these older areas.

What we can expect from the study of a particular problem using functional analytic methods? First, these methods do not yield explicit numerical or analytical solutions. But they can confirm (or deny) the applicability of some numerical solution methods. They can also help us establish certain qualitative properties of boundary value problems. A qualitative study should start by establishing the existence (or non-existence) of solutions depending on the classes to which the external parameters belong. Then follows an investigation into the uniqueness and, after that, the smoothness, of solutions. Next, continuity of the dependence of solutions on external parameters should be examined. This requires a viewpoint on functions as whole entities, and is closely related to notions of mechanical stability. We can also establish such system properties as periodicity of solutions and steady-state behavior. An important class of questions amenable to functional analysis centers on the spectra for various mechanical models. Here the situation differs greatly from that presented in standard linear algebra. So a researcher should be familiar with this circle of questions, at least.

Functional analysis promotes a viewpoint in which we regard the state of a body as a unified mathematical object. Here we begin to deal with transformations of internal parameters under changes in external load, etc., much as we do in linear algebra: by considering matrices as unified objects that operate on and transform vectors. Operators for boundary value problems of continuum mechanics are much more complex though, and it is instructive to see which results from linear algebra carry over without significant modification.

Even those who work in applications should understand the pitfalls of any given area. Sometimes an acute awareness of these comes to the fore when empirical tests performed on a design fail to match theoretically-based expectations. During such moments it is helpful to know how the trouble may have arisen — from an inadequate model, perhaps, or the use of an inappropriate numerical scheme. But it is certainly better to avoid this situation in advance through suitable modifications made to standardized computer code. For example, if a careful qualitative study of a problem indicates that singularities may arise during its solution, then a method can be chosen to accommodate this (e.g., via asymptotic analysis done prior to the numerical computation stage). But this can happen only when the researcher is aware of such a possibility.

Functional analysis offers practical tools for the quantitative description of mechanics problems and their solutions — tools understandable only to those familiar with its terminology and main results. As time goes on it will continue to prove indispensable to anyone who wishes to do serious work in continuum mechanics and many other areas. As the old saying goes, a good theory is the most practical thing.

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