Appendix A

Formulary

For convenience we list here the main tensor formulas obtained in each chapter. Note: the symbol ∀ denotes the universal quantifier (read as "for all" or "for every").

Chapter 2

Reciprocal (dual) basis

Kronecker delta symbol

$$\delta^i_j = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases}$$

Definition of reciprocal basis

$$\mathbf{e}_j \cdot \mathbf{e}^i = \delta^i_j$$

Components of a vector

$$\mathbf{x} = x^i \mathbf{e}_i = x_i \mathbf{e}^i \qquad \begin{aligned} x^i &= \mathbf{x} \cdot \mathbf{e}^i \\ x_i &= \mathbf{x} \cdot \mathbf{e}_i \end{aligned}$$

Relations between dual bases

$$\mathbf{e}^i = \frac{1}{V}(\mathbf{e}_j \times \mathbf{e}_k) \qquad \mathbf{e}_i = \frac{1}{V'}(\mathbf{e}^j \times \mathbf{e}^k)$$

where

$$(i, j, k) = (1, 2, 3) \text{ or } (2, 3, 1) \text{ or } (3, 1, 2)$$

and

$$\begin{aligned} V &= \mathbf{e}_1 \cdot (\mathbf{e}_2 \times \mathbf{e}_3) \\ V' &= \mathbf{e}^1 \cdot (\mathbf{e}^2 \times \mathbf{e}^3) \end{aligned} \qquad V' = 1/V$$

Metric coefficients

$$\begin{aligned} g^{jq} &= \mathbf{e}^j \cdot \mathbf{e}^q \\ g_{ip} &= \mathbf{e}_i \cdot \mathbf{e}_p \end{aligned} \qquad g_{ij} g^{jk} = \delta_i^k$$

In Cartesian frames,

$$g_{ij} = \delta_i^j \qquad g^{ij} = \delta_j^i$$

Dot products in mixed and unmixed bases

$$\mathbf{a} \cdot \mathbf{b} = a^i b^j g_{ij} = a_i b_j g^{ij} = a^i b_i = a_i b^i$$

Raising and lowering of indices

$$x_j = x^i g_{ij} \qquad x^i = x_j g^{ij}$$

Frame transformation

Equations of transformation

$$\begin{aligned} \mathbf{e}_i &= A_i^j \tilde{\mathbf{e}}_j & A_i^j &= \mathbf{e}_i \cdot \tilde{\mathbf{e}}^j \\ \tilde{\mathbf{e}}_i &= \tilde{A}_i^j \mathbf{e}_j & \tilde{A}_i^j &= \tilde{\mathbf{e}}_i \cdot \mathbf{e}^j \end{aligned}$$

where

$$\tilde{A}_i^j A_j^k = A_i^j \tilde{A}_j^k = \delta_i^k$$

Vector components and transformation laws

$$\mathbf{x} = x^i \mathbf{e}_i = x_i \mathbf{e}^i = \tilde{x}^i \tilde{\mathbf{e}}_i = \tilde{x}_i \tilde{\mathbf{e}}^i$$

and

$$\tilde{x}^i = A^i_j x^j \qquad x^i = \tilde{A}^i_j \tilde{x}^j$$
$$\tilde{x}_i = \tilde{A}^j_i x_j \qquad x_i = A^j_i \tilde{x}_j$$

Miscellaneous

Permutation (Levi–Civita) symbol

$$\epsilon_{ijk} = \mathbf{e}_i \cdot (\mathbf{e}_j \times \mathbf{e}_k) = \begin{cases} +V & (i,j,k) \text{ an even permutation of } (1,2,3) \\ -V & (i,j,k) \text{ an odd permutation of } (1,2,3) \\ 0 & \text{two or more indices equal} \end{cases}$$
$$\epsilon^{ijk} = \mathbf{e}^i \cdot (\mathbf{e}^j \times \mathbf{e}^k) = \begin{cases} +V' & (i,j,k) \text{ an even permutation of } (1,2,3) \\ -V' & (i,j,k) \text{ an odd permutation of } (1,2,3) \\ 0 & \text{two or more indices equal} \end{cases}$$

Useful identities

$$\epsilon_{ijk}\epsilon^{pqr} = \begin{vmatrix} \delta^p_i & \delta^q_i & \delta^r_i \\ \delta^p_j & \delta^q_j & \delta^r_j \\ \delta^p_k & \delta^q_k & \delta^r_k \end{vmatrix} \qquad \epsilon_{ijk}\epsilon^{pqk} = \delta^p_i \delta^q_j - \delta^q_i \delta^p_j$$

Determinant of Gram matrix

$$V^2 = \det[g_{ij}]$$

Cross product

$$\mathbf{a} \times \mathbf{b} = \mathbf{e}^i \epsilon_{ijk} a^j b^k = \mathbf{e}_i \epsilon^{ijk} a_j b_k$$

Chapter 3

Dyad product

Properties

$$(\lambda\mathbf{a}) \otimes \mathbf{b} = \mathbf{a} \otimes (\lambda\mathbf{b}) = \lambda(\mathbf{a} \otimes \mathbf{b})$$
$$(\mathbf{a} + \mathbf{b}) \otimes \mathbf{c} = \mathbf{a} \otimes \mathbf{c} + \mathbf{b} \otimes \mathbf{c}$$
$$\mathbf{a} \otimes (\mathbf{b} + \mathbf{c}) = \mathbf{a} \otimes \mathbf{b} + \mathbf{a} \otimes \mathbf{c}$$

Dot products of dyad with vector

$$\mathbf{ab} \cdot \mathbf{c} = (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$$
$$\mathbf{c} \cdot (\mathbf{ab}) = (\mathbf{c} \cdot \mathbf{a})\mathbf{b}$$

Tensors from operator viewpoint

Equality of tensors

$$\mathbf{A} = \mathbf{B} \iff \forall \mathbf{x},\ \mathbf{A} \cdot \mathbf{x} = \mathbf{B} \cdot \mathbf{x}$$

Components

$$a^{ij} = \mathbf{e}^i \cdot \mathbf{A} \cdot \mathbf{e}^j$$
$$a_{ij} = \mathbf{e}_i \cdot \mathbf{A} \cdot \mathbf{e}_j$$
$$a^{i}_{\cdot j} = \mathbf{e}^i \cdot \mathbf{A} \cdot \mathbf{e}_j$$
$$a_{i}^{\cdot j} = \mathbf{e}_i \cdot \mathbf{A} \cdot \mathbf{e}^j$$

Definition of sum $\mathbf{A} + \mathbf{B}$

$$\forall \mathbf{x},\ (\mathbf{A} + \mathbf{B}) \cdot \mathbf{x} = \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{x}$$

Definition of scalar multiple $c\mathbf{A}$

$$\forall \mathbf{x},\ (c\mathbf{A}) \cdot \mathbf{x} = c(\mathbf{A} \cdot \mathbf{x})$$

Definition of dot product $\mathbf{A}\cdot\mathbf{B}$

$$(\mathbf{A}\cdot\mathbf{B})\cdot\mathbf{x} \equiv \mathbf{A}\cdot(\mathbf{B}\cdot\mathbf{x})$$

Definition of pre-multiplication $\mathbf{y}\cdot\mathbf{A}$

$$\forall\mathbf{x},\ (\mathbf{y}\cdot\mathbf{A})\cdot\mathbf{x} = \mathbf{y}\cdot(\mathbf{A}\cdot\mathbf{x})$$

Definition of unit tensor $\mathbf{E}$

$$\forall\mathbf{x},\ \mathbf{E}\cdot\mathbf{x} = \mathbf{x}\cdot\mathbf{E} = \mathbf{x}$$

Unit tensor components

$$\mathbf{E} = \mathbf{e}^i\mathbf{e}_i = \mathbf{e}_j\mathbf{e}^j = g_{ij}\mathbf{e}^i\mathbf{e}^j = g^{ij}\mathbf{e}_i\mathbf{e}_j$$

Inverse tensor

$$\mathbf{A}\cdot\mathbf{A}^{-1} = \mathbf{E}$$

$$(\mathbf{A}\cdot\mathbf{B})^{-1} = \mathbf{B}^{-1}\cdot\mathbf{A}^{-1}$$

Nonsingular tensor $\mathbf{A}$

$$\mathbf{A}\cdot\mathbf{x} = 0 \implies \mathbf{x} = 0$$

Determinant of a tensor

$$\begin{aligned}\det\mathbf{A} &= \left|a_i^{\cdot j}\right| = \left|a^k_{\cdot m}\right| = \frac{1}{g}\left|a_{st}\right| = g\left|a^{pq}\right| \\ &= \frac{1}{6}\epsilon_{ijk}\epsilon^{mnp}a_m^{\cdot i}a_n^{\cdot j}a_p^{\cdot k}\end{aligned}$$

Dyadic components under transformation

Transformation to reciprocal basis

$$a_{km} = a^{ij}g_{ki}g_{jm}$$

More general transformation

$$\mathbf{e}_i = A_i^j \tilde{\mathbf{e}}_j \implies \tilde{a}^{ij} = a^{km} A_k^i A_m^j$$

$$\tilde{\mathbf{e}}_i = \tilde{A}_i^j \mathbf{e}_j \implies a^{ij} = \tilde{a}^{km} \tilde{A}_k^i \tilde{A}_m^j$$

$$A_j^k \tilde{A}_k^i = \delta_j^i$$

$$\begin{aligned} \mathbf{A} &= \tilde{a}^{ij} \tilde{\mathbf{e}}_i \tilde{\mathbf{e}}_j = \tilde{a}_{kl} \tilde{\mathbf{e}}^k \tilde{\mathbf{e}}^l = \tilde{a}_i^{\cdot j} \tilde{\mathbf{e}}^i \tilde{\mathbf{e}}_j = \tilde{a}_{\cdot l}^k \tilde{\mathbf{e}}_k \tilde{\mathbf{e}}^l \\ &= a^{ij} \mathbf{e}_i \mathbf{e}_j = a_{kl} \mathbf{e}^k \mathbf{e}^l = a_i^{\cdot j} \mathbf{e}^i \mathbf{e}_j = a_{\cdot l}^k \mathbf{e}_k \mathbf{e}^l \end{aligned}$$

where

$$\begin{aligned} \tilde{a}^{ij} &= A_k^i A_l^j a^{kl} & a^{ij} &= \tilde{A}_k^i \tilde{A}_l^j \tilde{a}^{kl} \\ \tilde{a}_{ij} &= \tilde{A}_i^k \tilde{A}_j^l a_{kl} & a_{ij} &= A_i^k A_j^l \tilde{a}_{kl} \\ \tilde{a}_{\cdot j}^i &= A_k^i \tilde{A}_j^l a_{\cdot l}^k & a_{\cdot j}^i &= \tilde{A}_k^i A_j^l \tilde{a}_{\cdot l}^k \\ \tilde{a}_i^{\cdot j} &= \tilde{A}_i^k A_l^j a_k^{\cdot l} & a_i^{\cdot j} &= A_i^k \tilde{A}_l^j \tilde{a}_k^{\cdot l} \end{aligned}$$

More dyadic operations

Dot product

$$\mathbf{ab} \cdot \mathbf{cd} = (\mathbf{b} \cdot \mathbf{c})\mathbf{ad}$$

$$\begin{aligned} \mathbf{A} \cdot (\lambda \mathbf{a} + \mu \mathbf{b}) &= \lambda \mathbf{A} \cdot \mathbf{a} + \mu \mathbf{A} \cdot \mathbf{b} \\ (\lambda \mathbf{A} + \mu \mathbf{B}) \cdot \mathbf{a} &= \lambda \mathbf{A} \cdot \mathbf{a} + \mu \mathbf{B} \cdot \mathbf{a} \end{aligned}$$

Double dot product

$$\mathbf{ab} \cdot\cdot\, \mathbf{cd} = (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{d})$$

Second rank tensor topics

Transpose

$$\mathbf{A}^T = a^{ji}\mathbf{e}_i\mathbf{e}_j = a_{ji}\mathbf{e}^i\mathbf{e}^j = a^j_{\cdot i}\mathbf{e}^i\mathbf{e}_j = a_j^{\cdot i}\mathbf{e}_i\mathbf{e}^j$$

$$\mathbf{A}\cdot\mathbf{x} = \mathbf{x}\cdot\mathbf{A}^T$$

$$(\mathbf{A}^T)^T = \mathbf{A}$$

$$(\mathbf{A}\cdot\mathbf{B})^T = \mathbf{B}^T\cdot\mathbf{A}^T$$

$$\mathbf{a}\cdot\mathbf{C}^T\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{C}\cdot\mathbf{a}$$

$$\det\mathbf{A}^{-1} = (\det\mathbf{A})^{-1} \qquad (\mathbf{A}\cdot\mathbf{B})^{-1} = \mathbf{B}^{-1}\cdot\mathbf{A}^{-1}$$
$$(\mathbf{A}^T)^{-1} = (\mathbf{A}^{-1})^T \qquad (\mathbf{A}^{-1})^{-1} = \mathbf{A}$$

Tensors raised to powers

$$\mathbf{A}^0 = \mathbf{A} \qquad \mathbf{A}^n = \mathbf{A}\cdot\mathbf{A}^{n-1} \text{ for } n = 1, 2, 3, \ldots$$

$$\mathbf{A}^{-n} = \mathbf{A}^{-n+1}\cdot\mathbf{A}^{-1} \text{ for } n = 2, 3, 4, \ldots$$

$$e^{\mathbf{A}} = \mathbf{E} + \frac{\mathbf{A}}{1!} + \frac{\mathbf{A}^2}{2!} + \frac{\mathbf{A}^3}{3!} + \cdots$$

Symmetric and antisymmetric tensors

$$\begin{array}{ll} \text{symmetric:} & \mathbf{A} = \mathbf{A}^T; \quad \forall \mathbf{x},\ \mathbf{A} \cdot \mathbf{x} = \mathbf{x} \cdot \mathbf{A} \\ \text{antisymmetric:} & \mathbf{A} = -\mathbf{A}^T \end{array}$$

$$\mathbf{A} = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^T\right) + \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^T\right)$$

Eigenpair

$$\mathbf{A} \cdot \mathbf{x} = \lambda \mathbf{x}$$

Viète formulas for invariants

$$\begin{aligned} I_1(\mathbf{A}) &= \lambda_1 + \lambda_2 + \lambda_3 && = \text{trace } \mathbf{A} \\ I_2(\mathbf{A}) &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 && \\ I_3(\mathbf{A}) &= \lambda_1\lambda_2\lambda_3 && = \det \mathbf{A} \end{aligned}$$

Orthogonal tensor

$$\mathbf{Q} \cdot \mathbf{Q}^T = \mathbf{Q}^T \cdot \mathbf{Q} = \mathbf{E}$$

Polar decompositions

$$\begin{array}{ll} \mathbf{A} = \mathbf{S} \cdot \mathbf{Q} & \mathbf{Q} \text{ orthogonal} \\ \mathbf{A} = \mathbf{Q} \cdot \mathbf{S}' & \mathbf{S}, \mathbf{S}' \text{ positive definite and symmetric} \end{array}$$

Chapter 4

Vector fields

Some rules for differentiating vector functions

$$\frac{d(\mathbf{e}_1(t)+\mathbf{e}_1(t))}{dt} = \frac{d\mathbf{e}_1(t)}{dt} + \frac{d\mathbf{e}_2(t)}{dt}$$
$$\frac{d(c\mathbf{e}_1(t))}{dt} = c\frac{d\mathbf{e}_1(t)}{dt}$$
$$\frac{d(f(t)\mathbf{e}(t))}{dt} = \frac{df(t)}{dt}\mathbf{e}(t) + f(t)\frac{d\mathbf{e}(t)}{dt}$$

$$\frac{d}{dt}(\mathbf{e}_1(t)\cdot\mathbf{e}_2(t)) = \mathbf{e}_1'(t)\cdot\mathbf{e}_2(t) + \mathbf{e}_1(t)\cdot\mathbf{e}_2'(t)$$
$$\frac{d}{dt}(\mathbf{e}_1(t)\times\mathbf{e}_2(t)) = \mathbf{e}_1'(t)\times\mathbf{e}_2(t) + \mathbf{e}_1(t)\times\mathbf{e}_2'(t)$$

and

$$\begin{aligned}\frac{d}{dt}[\mathbf{e}_1(t)\,\mathbf{e}_2(t)\,\mathbf{e}_3(t)] = {} & [\mathbf{e}_1'(t)\,\mathbf{e}_2(t)\,\mathbf{e}_3(t)] + \\ & + [\mathbf{e}_1(t)\,\mathbf{e}_2'(t)\,\mathbf{e}_3(t)] + \\ & + [\mathbf{e}_1(t)\,\mathbf{e}_2(t)\,\mathbf{e}_3'(t)]\end{aligned}$$

where

$$[\mathbf{e}_1(t)\,\mathbf{e}_2(t)\,\mathbf{e}_3(t)] = (\mathbf{e}_1(t)\times\mathbf{e}_2(t))\cdot\mathbf{e}_3(t)$$

Tangent vectors to coordinate lines

$$\mathbf{r}_i = \frac{\partial\mathbf{r}}{\partial q^i} \qquad i = 1,2,3$$

Jacobian

$$\sqrt{g} = \mathbf{r}_1\cdot(\mathbf{r}_2\times\mathbf{r}_3) = \left|\frac{\partial x^i}{\partial q^j}\right|$$

Pointwise definition of reciprocal basis

$$\mathbf{r}^i\cdot\mathbf{r}_j = \delta^i_j$$

Definition of metric coefficients

$$g_{ij} = \mathbf{r}_i \cdot \mathbf{r}_j$$
$$g^{ij} = \mathbf{r}^i \cdot \mathbf{r}^j$$
$$g_i^j = \mathbf{r}_i \cdot \mathbf{r}^j = \delta_i^j$$

Transformation laws

$$\mathbf{r}_i = A_i^j \tilde{\mathbf{r}}_j \qquad A_i^j = \frac{\partial \tilde{q}^j}{\partial q^i}$$

$$\tilde{\mathbf{r}}_i = \tilde{A}_i^j \mathbf{r}_j \qquad \tilde{A}_i^j = \frac{\partial q^j}{\partial \tilde{q}^i}$$

$$\tilde{f}^i = A_j^i f^j \qquad \tilde{f}_i = \tilde{A}_i^j f_j \qquad f^i = \tilde{A}_j^i \tilde{f}^j \qquad f_i = A_i^j \tilde{f}_j$$

Differentials and the nabla operator

Metric forms

$$(ds)^2 = d\mathbf{r} \cdot d\mathbf{r} = g_{ij} dq^i dq^j$$

Nabla operator

$$\nabla = \mathbf{r}^i \frac{\partial}{\partial q^i}$$

Gradient of a vector function

$$d\mathbf{f} = d\mathbf{r} \cdot \nabla \mathbf{f} = \nabla \mathbf{f}^T \cdot d\mathbf{r}$$

Divergence of vector

$$\operatorname{div} \mathbf{f} = \nabla \cdot \mathbf{f} = \mathbf{r}^i \cdot \frac{\partial \mathbf{f}}{\partial q^i}$$

Rotation and curl of vector

$$\operatorname{rot} \mathbf{f} = \nabla \times \mathbf{f} = \mathbf{r}^i \times \frac{\partial \mathbf{f}}{\partial q^i} \qquad \boldsymbol{\omega} = \frac{1}{2} \operatorname{rot} \mathbf{f}$$

Divergence and rotation of second-rank tensor

$$\nabla \cdot \mathbf{A} = \mathbf{r}^i \cdot \frac{\partial}{\partial q^i}\mathbf{A} \qquad \nabla \times \mathbf{A} = \mathbf{r}^i \times \frac{\partial}{\partial q^i}\mathbf{A}$$

Differentiation of a vector function

Christoffel coefficients of the second kind

$$\frac{\partial \mathbf{r}_i}{\partial q^j} = \Gamma_{ij}^k \mathbf{r}_k \qquad \frac{\partial \mathbf{r}^j}{\partial q^i} = -\Gamma_{it}^j \mathbf{r}^t$$

$$\Gamma_{ij}^k = \Gamma_{ji}^k$$

Christoffel coefficients of the first kind

$$\frac{1}{2}\left(\frac{\partial g_{it}}{\partial q^j} + \frac{\partial g_{tj}}{\partial q^i} - \frac{\partial g_{ji}}{\partial q^t}\right) = \Gamma_{ijt}$$

$$\Gamma_{ijk} = \Gamma_{jik}$$

Covariant differentiation

$$\frac{\partial \mathbf{f}}{\partial q^i} = \mathbf{r}^k \nabla_i f_k = \mathbf{r}_j \nabla_i f^j$$

$$\nabla_k f_i = \frac{\partial f_i}{\partial q^k} - \Gamma_{ki}^j f_j \qquad \nabla_k f^i = \frac{\partial f^i}{\partial q^k} + \Gamma_{kt}^i f^t$$

$$\nabla \mathbf{f} = \mathbf{r}^i \mathbf{r}^j \nabla_i f_j = \mathbf{r}^i \mathbf{r}_j \nabla_i f^j$$

Covariant differentiation of second-rank tensor

$$\frac{\partial}{\partial q^k}\mathbf{A} = \nabla_k a^{ij} \mathbf{r}_i \mathbf{r}_j = \nabla_k a_{ij} \mathbf{r}^i \mathbf{r}^j = \nabla_k a_i^{\cdot j} \mathbf{r}^i \mathbf{r}_j = \nabla_k a_{\cdot j}^{i} \mathbf{r}_i \mathbf{r}^j$$

$$\nabla_k a^{ij} = \frac{\partial a^{ij}}{\partial q^k} + \Gamma^i_{ks} a^{sj} + \Gamma^j_{ks} a^{is} \qquad \nabla_k a_{ij} = \frac{\partial a_{ij}}{\partial q^k} - \Gamma^s_{ki} a_{sj} - \Gamma^s_{kj} a_{is}$$

$$\nabla_k a_i^{\cdot j} = \frac{\partial a_i^{\cdot j}}{\partial q^k} - \Gamma^s_{ki} a_s^{\cdot j} + \Gamma^j_{ks} a_i^{\cdot s} \qquad \nabla_k a^i_{\cdot j} = \frac{\partial a^i_{\cdot j}}{\partial q^k} + \Gamma^i_{ks} a^s_{\cdot j} - \Gamma^s_{kj} a^i_{\cdot s}$$

Differential operations

$$\nabla \times \mathbf{f} = \mathbf{r}_k \epsilon^{ijk} \frac{\partial f_j}{\partial q^i}$$

$$\Gamma^i_{in} = \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial q^n}$$

$$\nabla \cdot \mathbf{f} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^i} \left(\sqrt{g} f^i \right)$$

$$\nabla \times \mathbf{A} = \epsilon^{kin} \mathbf{r}_n \frac{\partial}{\partial q^k} \left(\mathbf{r}^j a_{ij} \right)$$

$$\nabla \cdot \mathbf{A} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial q^i} \left(\sqrt{g} a^{ij} \mathbf{r}_j \right)$$

$$\nabla^2 f \equiv \nabla \cdot \nabla f = g^{ij} \left(\frac{\partial^2 f}{\partial q^i \partial q^j} - \Gamma^k_{ij} \frac{\partial f}{\partial q^k} \right)$$

$$\nabla^2 f = \nabla^j \nabla_j f, \qquad \nabla^j \equiv g^{ij} \nabla_i$$

$$\nabla^2 \mathbf{f} = \mathbf{r}_j \nabla^i \nabla_i f^j$$

$$\nabla \nabla \cdot \mathbf{f} = \mathbf{r}^i \nabla_i \nabla_j f^j$$

$$\nabla \times \nabla \times \mathbf{f} = \nabla \nabla \cdot \mathbf{f} - \nabla^2 \mathbf{f}$$

Orthogonal coordinate systems

Lamé coefficients

$$(H_i)^2 = g_{ii}$$
$$\mathbf{r}^i = \mathbf{r}_i/(H_i)^2 \qquad i = 1, 2, 3$$
$$\hat{\mathbf{r}}_i = \mathbf{r}_i/H_i$$

Differentiation in the orthogonal basis

$$\nabla = \frac{\hat{\mathbf{r}}_i}{H_i}\frac{\partial}{\partial q^i}$$

$$\nabla \mathbf{f} = \hat{\mathbf{r}}_i \hat{\mathbf{r}}_j \left(\frac{1}{H_i}\frac{\partial f_j}{\partial q^i} - \frac{f_i}{H_i H_j}\frac{\partial H_i}{\partial q^j} + \delta_{ij}\frac{f_k}{H_k}\frac{1}{H_i}\frac{\partial H_i}{\partial q^k} \right)$$

$$\nabla \cdot \mathbf{f} = \frac{1}{H_1 H_2 H_3}\left(\frac{\partial}{\partial q^1}(H_2 H_3 f_1) + \frac{\partial}{\partial q^2}(H_3 H_1 f_2) + \frac{\partial}{\partial q^3}(H_1 H_2 f_3) \right)$$

$$\nabla \times \mathbf{f} = \frac{1}{2}\frac{\hat{\mathbf{r}}_i \times \hat{\mathbf{r}}_j}{H_i H_j}\left(\frac{\partial}{\partial q^i}(H_j f_j) - \frac{\partial}{\partial q^j}(H_i f_i) \right)$$

$$\nabla^2 f = \frac{1}{H_1 H_2 H_3}\left[\frac{\partial}{\partial q^1}\left(\frac{H_2 H_3}{H_1}\frac{\partial f}{\partial q^1} \right) + \right.$$
$$+ \frac{\partial}{\partial q^2}\left(\frac{H_3 H_1}{H_2}\frac{\partial f}{\partial q^2} \right) +$$
$$\left. + \frac{\partial}{\partial q^3}\left(\frac{H_1 H_2}{H_3}\frac{\partial f}{\partial q^3} \right) \right]$$

Integration formulas

Transformation of multiple integral

$$\int_V f(x_1, x_2, x_3)\, dx_1\, dx_2\, dx_3 = \int_V f(q^1, q^2, q^3) J\, dq^1\, dq^2\, dq^3$$

$$J = \sqrt{g} = \left| \frac{\partial x_i}{\partial q^j} \right|$$

Integration by parts

$$\int_V \frac{\partial f}{\partial x_k} g \, dx_1 \, dx_2 \, dx_3 = -\int_V \frac{\partial g}{\partial x_k} f \, dx_1 \, dx_2 \, dx_3 + \int_S f g \, n_k \, dS$$

Miscellaneous results

$$\int_V \nabla f \, dV = \int_S f \mathbf{n} \, dS \qquad \int_V \nabla \cdot \mathbf{f} \, dV = \int_S \mathbf{n} \cdot \mathbf{f} \, dS$$
$$\int_V \nabla \mathbf{f} \, dV = \int_S \mathbf{n}\mathbf{f} \, dS \qquad \int_V \nabla \times \mathbf{f} \, dV = \int_S \mathbf{n} \times \mathbf{f} \, dS$$

$$\int_V \nabla \mathbf{A} \, dV = \int_S \mathbf{n}\mathbf{A} \, dS$$
$$\int_V \nabla \cdot \mathbf{A} \, dV = \int_S \mathbf{n} \cdot \mathbf{A} \, dS$$
$$\int_V \nabla \times \mathbf{A} \, dV = \int_S \mathbf{n} \times \mathbf{A} \, dS$$

$$\oint_\Gamma \mathbf{f} \cdot d\mathbf{r} = \int_S (\mathbf{n} \times \nabla) \cdot \mathbf{f} \, dS$$

$$\oint_\Gamma d\mathbf{r} \cdot \mathbf{A} = \int_S (\mathbf{n} \times \nabla) \cdot \mathbf{A} \, dS$$
$$\oint_\Gamma \mathbf{A} \cdot d\mathbf{r} = \int_S (\mathbf{n} \times \nabla) \cdot \mathbf{A}^T \, dS$$

Chapter 5

Elementary theory of curves

Parameterization

$$\mathbf{r} = \mathbf{r}(t) \qquad \text{or} \qquad \mathbf{r} = \mathbf{r}(s)$$

Length

$$s = \int_a^b |\mathbf{r}'(t)| \, dt$$

Unit tangent

$$\boldsymbol{\tau}(s) = \mathbf{r}'(s)$$

Equation of tangent line

$$\mathbf{r} = \mathbf{r}(t_0) + \lambda \mathbf{r}'(t_0)$$

$$\frac{x - x(t_0)}{x'(t_0)} = \frac{y - y(t_0)}{y'(t_0)} = \frac{z - z(t_0)}{z'(t_0)}$$

Curvature

$$k_1 = |\mathbf{r}''(s)| \qquad k_1^2 = \frac{(\mathbf{r}'(t) \times \mathbf{r}''(t))^2}{(\mathbf{r}'^2(t))^3}$$

Radius of curvature

$$R = 1/k_1$$

Principal normal, binormal

$$\boldsymbol{\nu} = \frac{\mathbf{r}''(s)}{k_1} \qquad \boldsymbol{\beta} = \boldsymbol{\tau} \times \boldsymbol{\nu}$$

Osculating plane

$$[\mathbf{r} - \mathbf{r}(s_0)] \cdot \boldsymbol{\beta}(s_0) = 0$$

$$\begin{vmatrix} x - x(t_0) & y - y(t_0) & z - z(t_0) \\ x'(t_0) & y'(t_0) & z'(t_0) \\ x''(t_0) & y''(t_0) & z''(t_0) \end{vmatrix} = 0$$

Torsion

$$k_2 = -\frac{(\mathbf{r}'(s) \times \mathbf{r}''(s)) \cdot \mathbf{r}'''(s)}{k_1^2} \qquad k_2 = -\frac{(\mathbf{r}'(t) \times \mathbf{r}''(t)) \cdot \mathbf{r}'''(t)}{(\mathbf{r}'(t) \times \mathbf{r}''(t))^2}$$

Serret–Frenet equations

$$\boldsymbol{\tau}' = k_1\boldsymbol{\nu}$$
$$\boldsymbol{\nu}' = -k_1\boldsymbol{\tau} - k_2\boldsymbol{\beta}$$
$$\boldsymbol{\beta}' = k_2\boldsymbol{\nu}$$

Theory of surfaces

Parameterization

$$\mathbf{r} = \mathbf{r}(u^1, u^2)$$

Tangent vectors, unit normal

$$\mathbf{r}_i = \frac{\partial \mathbf{r}}{\partial u^i} \qquad i = 1, 2$$

$$\mathbf{n} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{|\mathbf{r}_1 \times \mathbf{r}_2|}$$

First fundamental form

$$(ds)^2 = g_{ij}du^i du^j = E(du^1)^2 + 2F du^1 du^2 + G(du^2)^2$$

$$E = \mathbf{r}_1 \cdot \mathbf{r}_1 \qquad F = \mathbf{r}_1 \cdot \mathbf{r}_2 \qquad G = \mathbf{r}_2 \cdot \mathbf{r}_2$$

Orthogonality of curves

$$E du^1 d\tilde{u}^1 + F(du^1 d\tilde{u}^2 + du^2 d\tilde{u}^1) + G du^2 d\tilde{u}^2 = 0$$

Area

$$S = \int_A \sqrt{EG - F^2}\, du^1\, du^2$$

Second fundamental form

$$d^2\mathbf{r}\cdot\mathbf{n} = L(du^1)^2 + 2M\,du^1du^2 + N(du^2)^2 = -d\mathbf{r}\cdot d\mathbf{n}$$

$$L = \frac{\partial^2\mathbf{r}}{(\partial u^1)^2}\cdot\mathbf{n} \qquad M = \frac{\partial^2\mathbf{r}}{\partial u^1\partial u^2}\cdot\mathbf{n} \qquad N = \frac{\partial^2\mathbf{r}}{(\partial u^2)^2}\cdot\mathbf{n}$$

Normal curvature, mean curvature, Gaussian curvature

$$k_0 = k_1\cos\vartheta \qquad \vartheta = \text{ angle between } \boldsymbol{\nu} \text{ and } \mathbf{n}$$

$$H = \frac{1}{2}(k_{\min} + k_{\max}) = \frac{1}{2}\frac{LG - 2MF + NE}{EG - F^2}$$

$$K = k_{\min}k_{\max} = \frac{LN - M^2}{EG - F^2}$$

Surface given by $z = f(x,y)$ in Cartesian coordinates

Subscripts x, y denote partial derivatives with respect to x, y respectively.

$$E = 1 + {f_x}^2 \qquad F = f_x\,f_y \qquad G = 1 + {f_y}^2$$

$$EG - F^2 = 1 + {f_x}^2 + {f_y}^2 \qquad S = \int_D \sqrt{1 + {f_x}^2 + {f_y}^2}\,dx\,dy$$

$$\mathbf{n} = \frac{-f_x\mathbf{i}_1 - f_y\mathbf{i}_2 + \mathbf{i}_3}{\sqrt{1 + {f_x}^2 + {f_y}^2}}$$

$$L = \mathbf{r}_{xx} \cdot \mathbf{n} = \frac{f_{xx}}{\sqrt{1 + {f_x}^2 + {f_y}^2}}$$

$$M = \mathbf{r}_{xy} \cdot \mathbf{n} = \frac{f_{xy}}{\sqrt{1 + {f_x}^2 + {f_y}^2}}$$

$$N = \mathbf{r}_{yy} \cdot \mathbf{n} = \frac{f_{yy}}{\sqrt{1 + {f_x}^2 + {f_y}^2}}$$

$$K = \frac{f_{xx} f_{yy} - {f_{xy}}^2}{\left(1 + {f_x}^2 + {f_y}^2\right)^2}$$

Surface of revolution about z-axis

$$x = \phi(u) \qquad z = \psi(u)$$

$$(ds)^2 = \left({\phi'}^2 + {\psi'}^2\right) du^2 + \phi^2 \, dv^2$$

$$-d\mathbf{n} \cdot d\mathbf{r} = \frac{\psi'' \phi' - \phi'' \psi'}{\sqrt{{\phi'}^2 + {\psi'}^2}} \, du^2 + \frac{\psi' \phi}{\sqrt{{\phi'}^2 + {\psi'}^2}} \, dv^2$$