SEQUENCES AND SERIES

SEQUENCES

A list of numbers that follow some sort of pattern or rule is called a **sequence**. For example, $2, 4, 6, 8, \ldots$ is a sequence; the first term is 2, the second term is 4, and so on.

All sequences are either finite or infinite. A **finite sequence** has a last term. An **infinite sequence** has no last term and continues indefinitely (indicated by ...).

Each number in a sequence is called a term. The following notation is used to represent the terms in any sequence:

$$u_1, u_2, u_3, \ldots, u_n, \ldots$$

where

u_1 is the first term
u_2 is the second term
u_3 is the third term
$\vdots$ $\vdots$
u_n is the nth term

u_n is also called the general term.

For example, for the sequence

2, 4, 8, 16, 32, ...

we have

$u_1 = 2$ (The first term is 2)
$u_2 = 4$ (The second term is 4)
$u_3 = 8$ (The third term is 8)

and so on.

Often, a sequence can be described by a mathematical formula. The formula is a rule that gives us the value of any term. In fact, it gives us the value of the nth term, u_n, in terms of n.

EXAMPLE 1

Write down the first 3 terms of the sequence $u_1, u_2, u_3, \ldots$ if the nth term is given by the formula $u_n = n^2 + n$.

SOLUTION

We simply use the formula for the nth term:

Put $n = 1$ into the formula: $u_1 = 1^2 + 1 = 2$
Put $n = 2$ into the formula: $u_2 = 2^2 + 2 = 6$
Put $n = 3$ into the formula: $u_3 = 3^2 + 3 = 12$

Sometimes we are given a sequence and asked to find a formula for the *n*th term.

EXAMPLE 2

For the sequence 1, 4, 9, 16, ..., find a formula for the *n*th term.

SOLUTION

By inspection,

$$u_1 = 1 = 1^2$$
$$u_2 = 4 = 2^2$$
$$u_3 = 9 = 3^2$$
$$u_4 = 16 = 4^2$$

For the *n*th term, $u_n = n^2$.

EXAMPLE 3

For the following sequences, find the value of the tenth term.

(a) $u_n = \dfrac{3}{(n+1)(n+2)}$ (b) $u_n = \pi$

SOLUTION

Put $n = 10$ into the formula for u_n:

(a) $u_{10} = \dfrac{3}{10.11} = \dfrac{3}{110}$ (b) $u_{10} = \pi$

ARITHMETIC SEQUENCES

In an **arithmetic sequence**, each term is equal to the previous term plus a constant value.

For example, the sequence

$$5, 8, 11, 14, \ldots$$

is an arithmetic sequence. Here, each term is equal to the previous term plus the constant 3. In all arithmetic sequences the difference between adjacent terms is a constant value. This difference is called the **common difference**.

More examples:

Arithmetic Sequence	First Term	Common Difference
$5, 11, 17, 23, \ldots$	15	6
$12, 8, 4, 0, \ldots$	12	-4
$-3, 0, 3, 6, \ldots$	-3	3

If a is the first term of an arithmetic sequence and d is the common difference, then the sequence is

$$a,\ a+d,\ a+2d,\ a+3d,\ \ldots$$

Each term is d more than the previous term.

> In an arithmetic sequence, the value of the nth term is given by
>
> $$u_n = a + (n-1)d$$

EXAMPLE 4

For the sequence

2, 5, 8, 11, 14, ...

(a) Find a formula for the nth term

(b) Find the 35th term

SOLUTION

(a) This is an arithmetic sequence with a = 2 and d = 3.

$$\begin{aligned} u_n &= a + (n-1)d \\ &= 2 + (n-1).3 \end{aligned}$$

Thus,

$$u_n = 3n - 1$$

(b) Put n = 35 into the formula for the nth term: $u_{35} = 3 \times 35 - 1 = 104$.

GEOMETRIC SEQUENCES

In **geometric sequences**, each term is equal to the previous term *multiplied by* a constant value. For example, the sequence

2, 6, 18, 54, ...

is a geometric sequence since each term is 3 times the previous term. In all geometric sequences, the ratio of any term to its previous term is a constant value. This constant is called the **common ratio**.

For example,

Geometric Sequence	First Term	Common Ratio
1, 3, 9, 27, ...	1	3
$2, -6, 18, -24, \ldots$	2	-3
$-2, -1, -\frac{1}{2}, -\frac{1}{4}, \ldots$	-2	$\frac{1}{2}$

If the first term of a geometric sequence is a and the common ratio is r, the sequence is

$$a, ar, ar^2, ar^3, \ldots$$

Each term is r times the previous term.

> In a geometric sequence, the value of the nth term is given by
>
> $$u_n = ar^{n-1}$$

<u>EXAMPLE 5</u>

Find the twelfth term of the geometric sequence

$$2, 6, 18, 54, \ldots$$

<u>SOLUTION</u>

This is a geometric sequence since with $a = 2$ and $r = 3$. Therefore,

$$\begin{aligned} u_n &= ar^{n-1} \\ &= 2\times 3^{n-1} \end{aligned}$$

Put $n = 12$ into this formula for the nth term: $u_{12} = 2\times 3^{11} = 354\,294$.

<u>SERIES</u>

In mathematics, a series is the sum of terms of a sequence. For example,

4, 7, 10, 13, 16

is a sequence, while

4 + 7+ 10 + 13 + 16

is a series.

In general, if $u_1, u_2, u_3, \ldots$ is a sequence, then

$$u_1 + u_2 + u_3 + \ldots$$

is the (corresponding) series.

> A series is the sum of a sequence.

<u>ARITHMETIC SERIES</u>

An arithmetic series is the sum of an arithmetic sequence. Remember that an arithmetic sequence is of the form

$$a, a + d, a + 2d, \ldots, a + (n - 1)d, \ldots$$

The sum to n terms is

$$S_n = a \ + \ a + d \ + \ a + 2d \ + \ldots a + (n-1)d$$

We can use the following formula to evaluate this sum.

For an arithmetic series, the sum to n terms is

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$

or

$$S_n = \frac{n}{2}(a + l)$$

where l is the nth term

EXAMPLE 6

Find a formula for the sum to n terms of the series $5 + 1 - 3 - \cdots$ and hence find the sum to 20 terms.

SOLUTION

We have $a = 5$, $d = -4$. Therefore,

$$\begin{aligned} S_n &= \frac{n}{2}\left(2a + (n-1)d\right) \\ &= \frac{n}{2}\left(2\times 5 + (n-1)\times(-4)\right) \\ &= \frac{n}{2}(10 - 4n + 4) \\ &= \frac{n}{2}(14 - 4n) \\ &= n(7 - n) \end{aligned}$$

The sum to 10 terms is thus −260.

GEOMETRIC SERIES

A geometric series is the sum of a geometric sequence. Since a geometric sequence is of the form

$$a, ar, ar^2, ar^3, ..., ar^{n-1}, ...$$

the sum to n terms is

$$S_n = a + ar + ar^2 + \ldots ar^{n-1}$$

For a geometric series, the sum to n terms is

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

or

$$S_n = \frac{a - rl}{1 - r}$$

where l is the last term.

These formulae do not apply when $r = 1$. In this case, $S_n = na$.

EXAMPLE 7

Find the formula for the sum to n terms of the geometric series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$ and hence find the sum to 10 terms.

SOLUTION

For this series, $a=1$ and $r=\frac{1}{2}$. Therefore,

$$\begin{aligned} S_n &= \frac{a\left(1-r^n\right)}{1-r} \\ &= \frac{1\left(1-\left(\tfrac{1}{2}\right)^n\right)}{1-\tfrac{1}{2}} \\ &= 2\left(1-\tfrac{1}{2^n}\right) \\ &= 2-\tfrac{1}{2^{n-1}} \end{aligned}$$

Putting $n=10$ gives $S_{10}=2-\frac{1}{2^9}=\frac{1023}{512}$.

INFINITE GEOMETRIC SERIES

An infinite series has an infinite number of terms. The series

$$a+ar+ar^2+\ldots ar^{n-1}+\ldots$$

is an infinite geometric series where the first term is a and the common ratio is r.

LIMITING SUM

If the sum to n terms of an infinite geometric series approaches some value as the number of terms increases, then the series is said to have a limiting sum.

An infinite geometric series has a limiting sum if and only if $-1<r<1$. The limiting sum is then given by:

$$\begin{aligned} S &= a+ar+ar^2+\ldots \\ &= \frac{a}{1-r}, \quad \text{if and only if } -1<r<1 \end{aligned}$$

EXAMPLE 8

Find the limiting sum of the series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$

SOLUTION

This is an infinite geometric series with $a=1$ and $r=\frac{1}{2}$. Since $-1<r<1$, the series does have a limiting sum. Thus,

$$\begin{aligned} S &= 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots \\ &= \frac{1}{1-\frac{1}{2}} \\ &= 2 \end{aligned}$$

The following table gives the value of the first 10 terms and their sums for the series in the above example.

n	nth term	Sum to n terms
1	1	1
2	$\frac{1}{2}=0.5$	1.5
3	$\frac{1}{4}=0.25$	1.75
4	$\frac{1}{8}=0.125$	1.875
5	$\frac{1}{16}=0.0625$	1.9375
6	$\frac{1}{32}=0.03125$	1.96875
7	$\frac{1}{64}=0.015625$	1.984375
8	$\frac{1}{128}=0.0078125$	1.9921875
9	$\frac{1}{256}=0.00390625$	1.99609375
10	$\frac{1}{512}=0.001953125$	1.998046875

The table confirms that the series $S=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$ has a limiting sum of 2, and also that the nth term decreases in value (approaching 0) as n increases (approaching infinity).

EXAMPLE 9

Write an expression for $1-x+x^2-x^3+\ldots$ where $-1<x<1$.

SOLUTION

We have $a=1$ and $r=-x$. Since $-1<r<1$, a limiting sum exists:

$$
\begin{aligned}
S &= 1 - x + x^2 - x^3 + \dots \\
&= \frac{1}{1-(-x)} \\
&= \frac{1}{1+x}
\end{aligned}
$$

<u>EXAMPLE 10</u>

Explain why the series $1 + 1.1 + (1.1)^2 + (1.1)^3 + \dots$ does not have a limiting sum

<u>SOLUTION</u>

The common ratio for this series is $r = 1.1$ and since $r > 1$, the series does not have a limiting sum.