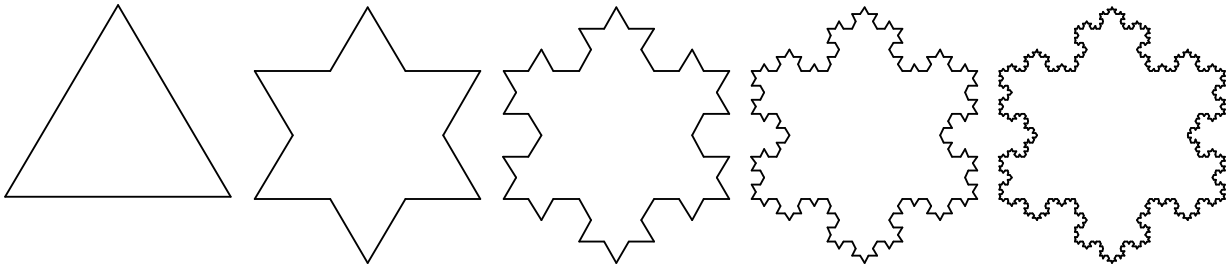
KOCH SNOWFLAKE

COURSE/LEVEL: *NSW Secondary High School Stage 5 Mathematics – Additional Content*

The Koch snowflake is created in stages. Starting with an equilateral triangle, new equilateral triangles are attached to the middle third of each side. New equilateral triangles are then added to each side of this shape, and then to this new shape, and so on, leading to the von Koch snowflake*:



Stage 0 Stage 1 Stage 2 Stage 3 Stage 4

1. At the *n*th stage of iteration of the Koch snowflake, I_n equals the number of sides, L_n equals the side length and P_n is its perimeter.

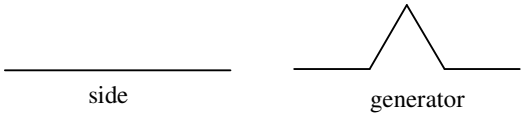
   (a) Explain the following, for $n = 1, 2, \ldots$,

   (i) $I_n = 4 \times I_{n-1}$ (ii) $L_n = \frac{L_{n-1}}{3}$ (iii) $P_n = \frac{4}{3} P_{n-1}$

   (b) Use these results to complete the table below. (Let *l* be the side length of the equilateral triangle at Stage 0).

Stage	Number of Sides	Side Length	Perimeter
0	3	l	P_0
1	4×3	$\frac{l}{3}$	$\frac{4}{3} \times P_0$
2	$4^2 \times 3$	$\frac{l}{3^2}$	$\left(\frac{4}{3}\right)^2 \times P_0$
3			
4			
n			

* Note: At each stage of iteration, each side is replaced by a "generator", illustrated below.



side generator

(c) Hence explain why the perimeter of the Koch snowflake approaches infinity as the number of stages increases.

2. (a) Complete the following table.

Stage	Number of sides	Number of new triangles	Area of each new triangle	Increase in area
0	3	0	A_0	A_0
1	4×3	3	$\frac{A_0}{9}$	$\frac{A_0}{3}$
2	$4^2\times3$	4×3	$\frac{A_0}{9^2}$	$\frac{4}{9}\times\frac{A_0}{3}$
3	$4^3\times3$	$4^2\times3$	$\frac{A_0}{9^3}$	$\left(\frac{4}{9}\right)^2\times\frac{A_0}{3}$
4				
5				
n				

(b) Let A_n be the area of the Koch snowflake at the nth stage of iteration. Write down an expression for A_n as a sum of n terms taken from the table.

(c) Let A be the area of the snowflake as n tends to infinity. Using the formula

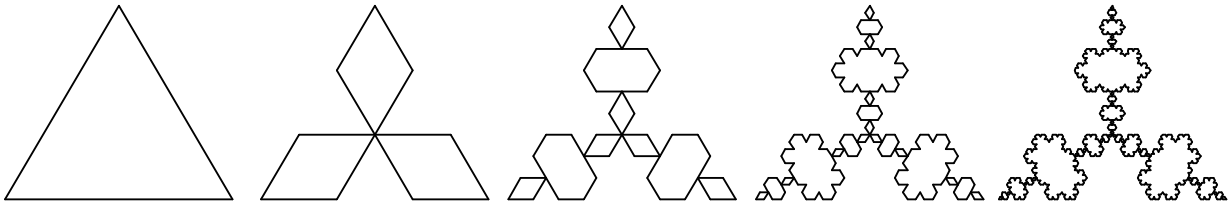
$$S = a + ar + ar^2 + \ldots = \frac{a}{1-r} \text{ (where } -1 < r < 1),$$

show that $A = \frac{8A_0}{5}$.

(d) Find an expression for A in terms of l.

KOCH ANTI-SNOWFLAKE

The Koch anti-snowflake is created from an equilateral triangle, as in the Koch snowflake. Here, equilateral triangles are removed from the middle third of each side at each iteration*.



Stage 0 Stage 1 Stage 2 Stage 3 Stage 4

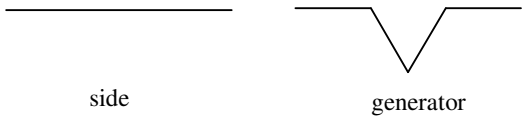
1. (a) Consider the snowflake at the *n*th stage of iteration and its corresponding anti-snowflake at the same stage of iteration.

   Explain why

   (i) The two shapes are equal in perimeter.
   (ii) The sum total area of the two shapes is equal to A_0, the area of the original equilateral triangle at Stage 0.

   (b) Hence show that the area of the Koch anti-snowflake (where *n* tends to infinity) is equal to $\frac{2A_0}{5}$.

* Note: At each stage of iteration, each side is replaced by a "generator", illustrated below.



side generator